

## A Mathematical Analysis on Linkage of a Network of Queues with Two Machines in a Flow Shop including Transportation Time

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**Abstract** This paper represents linkage network of queues consisting of biserial and parallel servers linked to a common server in series with a flowshop scheduling system consisting of two machines. The significant transportation time of the jobs from one machine to another is also considered. Further, the completion time of jobs/customers (waiting time + service time) in the queue network is the setup time for the first machine. The objective of the paper is of two folds, on one hand it minimizes the total waiting time and service time of jobs/customers in the queue network and on other hand it minimizes the idle time of the machines for the optimal sequence of jobs/customers in a given queue flowshop linkage model. A computer programme followed by a numerical illustration is given to justify the proposed algorithm.

**Keywords** Flowshop, Biserial, Waiting Time, Service Time, Makespan, Idle Time, Transportation Time.

### 1 Introduction

Waiting lines or queues are a common occurrence both in everyday life and in a variety of business and industrial situations. Forming a queue being a social phenomenon, it is beneficial to the society if it can be managed so that both the units that wait and the one that serves get the most benefit. The unit providing service is known as the server. Some examples are: In communication systems, voice or data traffic queue up for the lines for transmission, In a manufacturing system with several work stations, units completing work in one station wait for access to the next, Vehicles requiring service wait for their turn in a garage, Patients arrive at a doctor's clinic for treatment etc. Scheduling models concerned with the determination of an optimal sequence in which to service customers, or to perform a set of jobs, in order to minimize total elapsed time or another suitable measure of performance.

One of the earliest results in flow shop scheduling theory is an algorithm by Johnson's [1] for scheduling jobs in a two or three machine flow shop to minimize the time at which all jobs are completed. Jacksons [2] studied queuing systems with phase type service. Little's [3] derived the formula for calculating the mean queue length. Maggu and Das [4] discussed the effect of independent transportation time on scheduling of jobs. Maggu [5] introduced the

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concept of bitendom in theory of queues. Singh, T.P. [6] discussed network of queuing and scheduling system. Singh *et al* [7] studied the transient behavior of a queuing network with parallel biserial queues. Gupta *et al* [8] studied network of queues model comprised of Biserial and parallel channels linked with a common server. Kumar *et al* [9] discussed the steady state behavior of a queue model comprised of two subsystems with biserial channels linked with a common channel. As per literature only few efforts has been made to establish linkage between a network of queues and flowshop scheduling models. Maggu and kumar [10] introduced linkage between serial queuing and scheduling system. Singh and Kumar [11] studied linkage of queues in semi-series to a flowshop scheduling system. Singh *et al* [12] discussed the linkage of scheduling system with a serial queue network. Singh and Kumar [13] established linkage of a scheduling system with a biserial queue network. This paper combines the study of network of queues for providing the phase service in series with the flowshop network of two machines in a given order for processing the jobs.

Recently Gupta *et al* [14] made an attempt to link a network of queues consisting of a system of parallel biserial servers and a system of two parallel servers linked with a common server to a flow shop scheduling model. This paper is an attempt to extend their work by introducing the concept of independent transportation time, .i.e. the moving time for a job from one machine to another machine in the processing of jobs. This situation can be visualized when the machines on which jobs are to be processed are planted at different places and these jobs require additional times in their transplantation from one machine to another in the forms of loading time of jobs, moving time of jobs and then unloading time of jobs. The various queue characteristics have been obtained explicitly under steady state behavior of the system.

The paper is organized as follows: the second two is devoted to the mathematical model in which the queuing scheduling linkage model is explained. The section three is explored to deal with mathematical analysis of the proposed linkage model. The various queue characteristics are also derived in this section. The section four deals with various assumptions made along with the theorem to find the optimal sequence of jobs processing with significant transportation time. The section five is devoted to the algorithm proposed for optimizing the total flow time, average waiting time and mean service rate for the proposed linkage model.

## 2 Mathematical Model

The entire model is comprised of three servers  $S_1$ ,  $S_2$ ,  $S_3$  which is further linked with two machines  $M_1$  and  $M_2$  in series. The server  $S_1$  consists of two biserial service servers  $S_{11}$  and  $S_{12}$ . The server  $S_2$  contains two parallel servers  $S_{21}$  and  $S_{22}$ . Server  $S_3$  is commonly linked in series with each of two servers  $S_1$  and  $S_2$  for completion of first phase service demanded either at a subsystem  $S_1$  or  $S_2$ . The service time at  $S_{ij}$  ( $i, j=1, 2$ ) are exponentially distributed. Let mean service rate at  $S_{ij}$  ( $i, j=1, 2$ ) be  $\mu_1, \mu_2, \mu'_1, \mu'_2$  and  $\mu_3$  at  $S_3$  respectively. Queues  $Q_1$ ,  $Q_2$ ,  $Q_3$ ,  $Q_4$  and  $Q_5$  are said to be formed in front of the servers if they are busy. Customers coming at rate  $\lambda_1$  after completion of service at  $S_{11}$  will go to the network of the servers  $S_{11} \rightarrow S_{12}$  or  $S_{11} \rightarrow S_3$  with probabilities  $p_{12}$  or  $p_{13}$  such that  $p_{12} + p_{13} = 1$ . Further Customers coming at rate  $\lambda_2$  after completion of service at  $S_{12}$  will go to the network of the servers  $S_{12} \rightarrow S_{11}$  or  $S_{12} \rightarrow S_3$  with probabilities  $p_{21}$  or  $p_{23}$  such that  $p_{21} + p_{23} = 1$ . The completion time (waiting time + service time) of customers/jobs through queues  $Q_1$ ,  $Q_2$ ,  $Q_3$ ,  $Q_4$  and  $Q_5$  form

the setup time for the machine  $M_1$ . Let  $t_i$  be the transportation time of the  $i^{th}$  job from machine  $M_1$  to  $M_2$ . After coming out from the phase I, customers/jobs proceed to machines  $M_1$  and  $M_2$  for processing in phase II with processing times  $A_{i1}$  and  $A_{i2}$ . The objective is to develop a heuristic algorithm to find an optimal sequence of jobs/customers with minimum makespan in this queuing – scheduling linkage model.

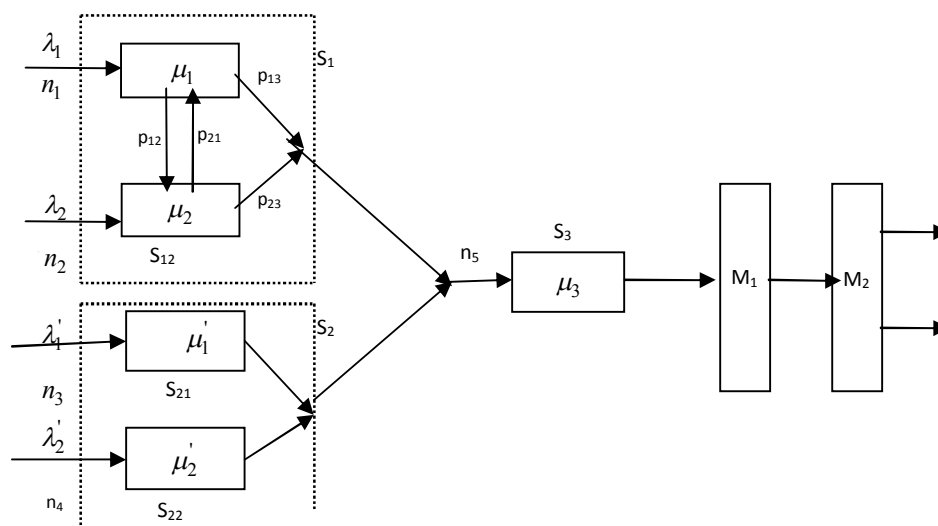


Fig. 1 Linkage Model

### 3 Mathematical Analysis

Let  $P_{n_1, n_2, n_3, n_4, n_5}$  be the joint probability that there are  $n_1$  units waiting in queue  $Q_1$  in front of  $S_{11}$ ,  $n_2$  units waiting in queue  $Q_2$  in front of  $S_{12}$ ,  $n_3$  units waiting in queue  $Q_3$  in front of  $S_{21}$ ,  $n_4$  units waiting in queue  $Q_4$  in front of  $S_{22}$ , and  $n_5$  units waiting in queue  $Q_5$  in front of  $S_3$  as shown in figure 1. In each case the waiting includes a unit in service, if any. Also,  $n_1, n_2, n_3, n_4, n_5 > 0$ .

The standard arguments lead to the following differential difference equations in transient form as

$$\begin{aligned} P_{n_1, n_2, n_3, n_4, n_5}(t) = & -(\lambda_1 + \lambda_2 + \lambda'_1 + \lambda'_2 + \mu_1 + \mu_2 + \mu'_1 + \mu'_2 + \mu_3)P_{n_1, n_2, n_3, n_4, n_5}(t) + \lambda_1 P_{n_1-1, n_2, n_3, n_4, n_5}(t) + \\ & \lambda_2 P_{n_1, n_2-1, n_3, n_4, n_5}(t) + \mu_1 (n_1 + 1) p_{13} P_{n_1+1, n_2, n_3, n_4, n_5-1}(t) + \\ & \mu_1 (n_1 + 1) p_{12} P_{n_1+1, n_2-1, n_3, n_4, n_5}(t) + \mu_2 (n_2 + 1) p_{23} P_{n_1, n_2+1, n_3, n_4, n_5-1}(t) + \\ & \mu_2 (n_2 + 1) p_{21} P_{n_1-1, n_2+1, n_3, n_4, n_5}(t) + \lambda'_1 P_{n_1, n_2, n_3-1, n_4, n_5}(t) + \lambda'_2 P_{n_1, n_2, n_3, n_4-1, n_5}(t) \\ & + \mu_3 (n_5 + 1) P_{n_1, n_2, n_3, n_4, n_5+1}(t) + \mu'_1 (n_3 + 1) P_{n_1, n_2, n_3+1, n_4, n_5-1}(t) + \mu'_2 (n_4 + 1) P_{n_1, n_2, n_3, n_4+1, n_5-1}(t). \end{aligned}$$

The steady state equation ( $t \rightarrow \infty$ ) governing the model are depicted as

$$\begin{aligned} (\lambda_1 + \lambda_2 + \lambda'_1 + \lambda'_2 + \mu_1 + \mu_2 + \mu'_1 + \mu'_2 + \mu_3)P_{n_1, n_2, n_3, n_4, n_5} = & \lambda_1 P_{n_1-1, n_2, n_3, n_4, n_5} + \\ & \lambda_2 P_{n_1, n_2-1, n_3, n_4, n_5} + \mu_1 (n_1 + 1) p_{13} P_{n_1+1, n_2, n_3, n_4, n_5-1} + \\ & \mu_1 (n_1 + 1) p_{12} P_{n_1+1, n_2-1, n_3, n_4, n_5} + \mu_2 (n_2 + 1) p_{23} P_{n_1, n_2+1, n_3, n_4, n_5-1} + \end{aligned}$$

$$\begin{aligned} & \mu_2(n_2 + 1)P_{21}P_{n_1-1, n_2+1, n_3, n_4, n_5} + \lambda_1' P_{n_1, n_2, n_3-1, n_4, n_5} + \lambda_2' P_{n_1, n_2, n_3, n_4-1, n_5} \\ & + \mu_3(n_5 + 1)P_{n_1, n_2, n_3, n_4, n_5+1} + \mu_1'(n_3 + 1)P_{n_1, n_2, n_3+1, n_4, n_5-1} + \mu_2'(n_4 + 1)P_{n_1, n_2, n_3, n_4+1, n_5-1}. \end{aligned} \quad (1)$$

Let us define the generating function as

$$F(X, Y, Z, R, S) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} P_{n_1, n_2, n_3, n_4, n_5} X^{n_1} Y^{n_2} Z^{n_3} R^{n_4} S^{n_5}$$

where

$$|X| = |Y| = |Z| = |R| = |S| = 1.$$

Also we define partial generating functions as

$$F_{n_2, n_3, n_4, n_5}(X) = \sum_{n_1=0}^{\infty} P_{n_1, n_2, n_3, n_4, n_5} X^{n_1}$$

$$F_{n_3, n_4, n_5}(X, Y) = \sum_{n_2=0}^{\infty} P_{n_2, n_3, n_4, n_5}(X) Y^{n_2}$$

$$F_{n_4, n_5}(X, Y, Z) = \sum_{n_3=0}^{\infty} P_{n_3, n_4, n_5}(X, Y) Z^{n_3}$$

$$F_{n_5}(X, Y, Z, R) = \sum_{n_4=0}^{\infty} P_{n_4, n_5}(X, Y, Z) R^{n_4}$$

$$F(X, Y, Z, R, S) = \sum_{n_5=0}^{\infty} P_{n_5}(X, Y, Z, R) S^{n_5}$$

Now, on taking  $n_1, n_2, n_3, n_4, n_5$  equal to zero one by one and then taking two of them pairwise, three of them at a time, four of them at a time and all of them; we get 32 equations. Now, on proceeding on the lines of Gupta *et al* [8] and following the standard technique, which after manipulation gives the final reduced result as:

$$\begin{aligned} & \mu_1 \left( 1 - \frac{S}{X} p_{13} - \frac{Y}{X} p_{12} \right) F(Y, Z, R, S) + \mu_2 \left( 1 - \frac{S}{Y} p_{23} - \frac{X}{Y} p_{21} \right) F(X, Z, R, S) + \\ & \mu_1' \left( 1 - \frac{S}{Z} \right) F(X, Y, R, S) + \mu_2' \left( 1 - \frac{S}{R} \right) F(X, Y, Z, S) + \mu_3 \left( 1 - \frac{1}{S} \right) F(X, Y, Z, R) \\ F(X, Y, Z, R, S) = & \frac{\lambda_1 (1 - X) + \lambda_2 (1 - Y) + \mu_1 \left( 1 - \frac{Y}{X} p_{12} - \frac{S}{X} p_{13} \right) + \mu_2 \left( 1 - \frac{S}{Y} p_{23} - \frac{X}{Y} p_{21} \right) + \lambda_1' (1 - Z) + \\ & \lambda_2' (1 - R) + \mu_1' \left( 1 - \frac{S}{Z} \right) + \mu_2' \left( 1 - \frac{S}{R} \right) + \mu_3 \left( 1 - \frac{1}{S} \right)}{\quad} \end{aligned} \quad (2)$$

For convenience, let us denote

$$\begin{aligned}
F(Y, Z, R, S) &= F_1 \\
F(X, Z, R, S) &= F_2 \\
F(X, Y, R, S) &= F_3 \\
F(X, Y, Z, S) &= F_4 \\
F(X, Y, Z, R) &= F_5
\end{aligned}$$

Also  $F(I, I, I, I) = I$ , the total probability.

On taking  $X = I$  as  $Y, Z, R, S \rightarrow 1$ ,  $F(X, Y, Z, R, S)$  is of  $\frac{0}{0}$  indeterminate form.

Now, on differentiating numerator and denominator of (2) separately w.r.t X, we have

$$\begin{aligned}
1 &= \frac{\mu_1(p_{13} + p_{12})F_1 + \mu_2(-p_{21})F_2}{-\lambda_1 + \mu_1(p_{12} + p_{13}) + \mu_2(-p_{21})} \\
\Rightarrow \mu_1 F_1 - \mu_2 p_{12} F_2 &= -\lambda_1 + \mu_1 - \mu_2 p_{21} \quad (\because p_{12} + p_{13} = 1)
\end{aligned} \tag{3}$$

Similarly, on differentiating numerator and denominator of (2) separately w.r.t Y, on taking  $Y = I$  and  $X, Z, R, S \rightarrow 1$  we have

$$\begin{aligned}
1 &= \frac{\mu_1(-p_{12})F_1 + \mu_2(p_{23} + p_{21})F_2}{-\lambda_2 + \mu_1(-p_{12}) + \mu_2(p_{23} + p_{21})} \\
\Rightarrow -\mu_1 p_{12} F_1 + \mu_2 F_2 &= -\lambda_2 - p_{12} \mu_1 + \mu_2 \quad (\because p_{23} + p_{21} = 1)
\end{aligned} \tag{4}$$

Again, on differentiating numerator and denominator of (2) separately w.r.t Z, on taking  $Z = 1$  and  $X, Y, R, S \rightarrow 1$  we have

$$1 = \frac{\mu_1' F_3}{-\lambda_1' + \mu_1'} \Rightarrow \mu_1' F_3 = -\lambda_1' + \mu_1' \tag{5}$$

Again, on differentiating numerator and denominator of (2) separately w.r.t R, on taking  $R = 1$  and  $X, Y, Z, S \rightarrow 1$  we have

$$1 = \frac{\mu_2' F_4}{-\lambda_2' + \mu_2'} \Rightarrow \mu_2' F_4 = -\lambda_2' + \mu_2' \tag{6}$$

Again, on differentiating numerator and denominator of (2) separately w.r.t S, on taking  $S = 1$  and  $X, Y, Z, R, S \rightarrow 1$  we have

$$\begin{aligned}
1 &= \frac{-\mu_1 p_{13} F_1 - \mu_2 p_{23} F_2 + \mu_1'(-F_3) + \mu_2'(-F_4) + \mu_3(F_5)}{\mu_1(-p_{13}) + \mu_2(-p_{23}) + \mu_1'(-1) + \mu_2'(-1) + \mu_3} \\
\Rightarrow -\mu_1 p_{13} F_1 - \mu_2 p_{23} F_2 - \mu_1' F_3 - \mu_2' F_4 + \mu_3 F_5 &= -p_{13} \mu_1 - p_{23} \mu_2 - \mu_1' - \mu_2' + \mu_3
\end{aligned} \tag{7}$$

On multiplying (4) with  $p_{21}$  and adding to (3), we get

$$F_1(\mu_1 - \mu_1 p_{12} p_{21}) = -\lambda_1 + \mu_1(1 - p_{12} p_{21}) - \lambda_2 p_{21}$$

$$\Rightarrow F_1 = 1 - \frac{\lambda_1 + \lambda_2 p_{21}}{(1 - p_{12} p_{21}) \mu_1} \quad (8)$$

$$F_3 = 1 - \frac{\lambda_1'}{\mu_1'} \quad (\text{Using (5)}) \quad (9)$$

$$F_4 = 1 - \frac{\lambda_2'}{\mu_2'} \quad (\text{Using (6)}) \quad (10)$$

$$F_5 = 1 - \frac{\lambda_3'}{\mu_3'} \quad (\text{Using (7)}) \quad (11)$$

On multiplying (3) with  $p_{12}$  and adding to (4), we get

$$\begin{aligned} \mu_2(1 - p_{12} p_{21}) F_2 &= -\lambda_2 - \lambda_1 p_{12} + \mu_2(p_{12} - p_{21}) + \mu_1(1 - p_{21} p_{12}) \\ F_2 &= 1 - \frac{\lambda_2 + \lambda_1 p_{12}}{(1 - p_{12} p_{21}) \mu_2} \end{aligned} \quad (12)$$

Now on putting the values of  $F_1, F_2, F_3, F_4$  in (8), we get

$$F_5 = 1 - \left[ \frac{\lambda_1' + \lambda_2'}{\mu_3'} + \frac{(\lambda_1 + \lambda_2 p_{12}) p_{13} + (\lambda_2 + \lambda_1 p_{12}) p_{23}}{\mu_3(1 - p_{12} p_{21})} \right] \quad (13)$$

On using the values of  $F_1, F_2, F_3, F_4$  and  $F_5$ , the joint probability is given by

$$P_{n_1, n_2, n_3, n_4, n_5} = \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \rho_4^{n_4} \rho_5^{n_5} (1 - \rho_1)(1 - \rho_2)(1 - \rho_3)(1 - \rho_4)(1 - \rho_5)$$

Where  $\rho_1 = 1 - F_1, \rho_2 = 1 - F_2, \rho_3 = 1 - F_3, \rho_4 = 1 - F_4, \rho_5 = 1 - F_5$ .

Further the solution in a steady state condition exist if  $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5 < 1$ .

### 3.1 Mean Queue Length

Average number of the customer (L)

$$\begin{aligned} &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} (n_1 + n_2 + n_3 + n_4 + n_5) P_{n_1, n_2, n_3, n_4, n_5} \\ &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} n_1 P_{n_1, n_2, n_3, n_4, n_5} + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} n_2 P_{n_1, n_2, n_3, n_4, n_5} + \dots \\ &+ \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} n_5 P_{n_1, n_2, n_3, n_4, n_5} \end{aligned}$$

Therefore

$$L = L_1 + L_2 + L_3 + L_4 + L_5$$

Further,

$$\begin{aligned}
 L_1 &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} n_1 P_{n_1, n_2, n_3, n_4, n_5} \\
 &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} n_1 \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \rho_4^{n_4} \rho_5^{n_5} (1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5) \\
 &= (1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5) \sum_{n_1=0}^{\infty} n_1 \rho_1^{n_1} \sum_{n_2=0}^{\infty} \rho_2^{n_2} \sum_{n_3=0}^{\infty} \rho_3^{n_3} \sum_{n_4=0}^{\infty} \rho_4^{n_4} \sum_{n_5=0}^{\infty} \rho_5^{n_5} \\
 &= \frac{\rho_1}{1-\rho_1}.
 \end{aligned}$$

Similarly

$$L_2 = \frac{\rho_2}{1-\rho_2}, L_3 = \frac{\rho_3}{1-\rho_3}, L_4 = \frac{\rho_4}{1-\rho_4}, L_5 = \frac{\rho_5}{1-\rho_5}.$$

Therefore, mean queue length = L

$$= L_1 + L_2 + L_3 + L_4 + L_5 = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + \frac{\rho_3}{1-\rho_3} + \frac{\rho_4}{1-\rho_4} + \frac{\rho_5}{1-\rho_5}$$

### 3.2 Average Waiting Time

The average waiting time and the average number of items waiting for a service in a service system are important measurements for a manager. Little's Law relates these two metrics via the average rate of arrivals to the system. This fundamental law has found numerous uses in operations management and managerial decision making. Little's Law says that, under steady state conditions, the average number of items in a queuing system equals the average rate at which items arrive multiplied by the average time that an item spends in the system. Let

L = Average number of items in the queuing system,

W = Average waiting time in the system for an item, and

A = Average number of items arriving per unit time.

By Little's formula, we have  $L = W\lambda$  ; or  $W = \frac{L}{\lambda}$ .

### 4 Assumptions, Theorem and Algorithm

The following assumptions are made for developing the proposed algorithm.

1. We assume that the arrival rate in the queue network follows Poisson distribution.
2. Each job/customer is processed on the machines  $M_1$  and  $M_2$  in the same order and pre-emption is not allowed, .i.e. once a job is started on a machine, the process on that machine cannot be stopped unless job is completed.
3. For the existence of the steady state behavior the following conditions hold good:

$$\begin{aligned}
(i) \rho_1 &= \frac{(\lambda_1 + \lambda_2 p_{21})}{\mu_1 (1 - p_{12} p_{21})} < 1 \\
(ii) \rho_2 &= \frac{(\lambda_2 + \lambda_1 p_{12})}{\mu_2 (1 - p_{12} p_{21})} < 1 \\
(iii) \rho_3 &= \frac{\lambda_1'}{\mu_1'} < 1 \\
(iv) \rho_4 &= \frac{\lambda_2'}{\mu_2'} < 1 \\
(vi) \rho_5 &= \left[ \frac{\lambda_1' + \lambda_2'}{\mu_3} + \frac{p_{13} (\lambda_1 + \lambda_2 p_{21}) + p_{23} (\lambda_2 + \lambda_1 p_{12})}{\mu_3 (1 - p_{12} p_{21})} \right] < 1
\end{aligned}$$

#### 4.1 Theorem.

Consider flowshop consisting of  $n$  jobs and two machines A and B. All jobs are to be processed on these machines according to the order AB and each machine can handle only one job at a time and each job  $i$  has transportation time  $t_i$  from machine A to machine B and they are known prior to making scheduling decisions. An optimal ordering of jobs to minimize total elapsed time is given by the following rule: job  $i$  proceeds job  $i+1$  if  $\min \{A_{i,A} + t_i, A_{i+1,B} + t_{i+1}\} < \min \{A_{i+1,A} + t_{i+1}, A_{i,B} + t_i\}$ .

**Proof.** Let  $S$  and  $S'$  be the sequences of jobs given by

$$\begin{aligned}
S &= J_1 - J_2 - J_3 - \dots - J_{i-1} - J_i - J_{i+1} - \dots - J_n \\
S' &= J_1 - J_2 - J_3 - \dots - J_{i-1} - J_{i+1} - J_i - \dots - J_n
\end{aligned}$$

Let  $(A_{p,x}, A'_{p,x})$  and  $(C_{p,x}, C'_{p,x})$  denotes the processing times and completion time of  $p^{th}$  job on machine  $x$  in the processes of sequence  $(S, S')$  respectively. Let  $(t_p, t'_p)$  denotes the transportation times of  $p^{th}$  job from machine A to the machine B in the processes of sequence  $(S, S')$  respectively.

$$\text{By definition, we have, } C_{p,B} = \max(C_{p,A} + t_p, C_{p-1,B}) + A_{p,B} \quad (1)$$

Now, sequence  $S$  is preferable to  $S'$  for  $n$  jobs if

$$C_{n,B} < C'_{n,B} \quad (2)$$

$$\text{i.e. } \max(C_{n,A} + t_n, C_{n-1,B}) + A_{n,B} < \max(C'_{n,A} + t'_n, C'_{n-1,B}) + A'_{n,B}$$

$$\text{Now, } C_{n,A} + t_n = \sum_{i=1}^n A_{i,A} + t_n = C'_{n,A} + t_n \quad C_{n,A} + t_n = \sum_{i=1}^n A_{i,A} + t_n = C'_{n,A} + t_n$$

Also,  $A_{n,B} = A'_{n,B}$

Inequality (2) will hold if  $C_{n-1,B} < C'_{n-1,B}$  (3)

Continuing in this way, one can easily get

$$C_{p,B} < C'_{p,B}, \quad (p = i+1, i+2, i+3, \dots, n)$$

and

$$C_{i+1,B} < C'_{i+1,B} \quad (4)$$

Now

$$\begin{aligned} C_{i+1,B} &= \max \{C_{i+1,A} + t_{i+1}, C_{i,B}\} + A_{i+1,B} = \max \{C_{i+1,A} + t_{i+1}, \max \{C_{i,A} + t_i, C_{i-1,B}\} + A_{i,B}\} + A_{i+1,B} \\ &= \max \{C_{i+1,A} + t_{i+1}, \max \{C_{i,A} + t_i + A_{i,B}, C_{i-1,B} + A_{i,B}\}\} + A_{i+1,B} \\ &= \max \{C_{i+1,A} + t_{i+1} + A_{i+1,B}, C_{i,A} + t_i + A_{i,B} + A_{i+1,B}, C_{i-1,B} + A_{i,B} + A_{i+1,B}\} \\ &= \max \{(C_{i-1,A} + A_{i,A} + A_{i+1,A} + t_{i+1} + A_{i+1,B}), (C_{i-1,A} + A_{i,A} + t_i + A_{i,B} + A_{i+1,B}), (C_{i-1,B} + A_{i,B} + A_{i+1,B})\} \end{aligned} \quad (5)$$

Similarly,

$$\begin{aligned} C'_{i+1,B} &= \max \{(C'_{i-1,A} + A'_{i,A} + A'_{i+1,A} + t'_{i+1} + A'_{i+1,B}), (C'_{i-1,A} + A'_{i,A} + t'_i + A'_{i,B} + A'_{i+1,B}), \\ &\quad (C'_{i-1,B} + A'_{i,B} + A'_{i+1,B})\} \end{aligned} \quad (6)$$

Further, on comparing sequences S and S', we have

$$C_{i-1,A} = C'_{i-1,A}; C_{i-1,B} = C'_{i-1,B}; A_{i,x} = A'_{i+1,x}; t_i = t'_{i+1}; A_{i+1,x} = A'_{i,x}; t_{i+1} = t'_i \quad (7)$$

On using results (5), (6) and (7), the result (4) can be written as

$$\begin{aligned} &\max \{(C_{i-1,A} + A_{i,A} + A_{i+1,A} + t_{i+1} + A_{i+1,B}), (C_{i-1,A} + A_{i,A} + t_i + A_{i,B} + A_{i+1,B}), (C_{i-1,B} + A_{i,B} + A_{i+1,B})\} \\ &< \max \{(C_{i-1,A} + A_{i+1,A} + A_{i,A} + t_i + A_{i,B}), (C_{i-1,A} + A_{i+1,A} + t_{i+1} + A_{i+1,B} + A_{i,B}), (C_{i-1,B} + A_{i+1,B} + A_{i,B})\} \\ \text{Or } &\max \{C_{i-1,A} + A_{i,A} + A_{i+1,A} + t_{i+1} + A_{i+1,B}, C_{i-1,A} + A_{i,A} + t_i + A_{i,B} + A_{i+1,B}\} \\ &< \max \{C_{i-1,A} + A_{i+1,A} + A_{i,A} + t_i + A_{i,B}, C_{i-1,A} + A_{i+1,A} + t_{i+1} + A_{i+1,B} + A_{i,B}\} \end{aligned}$$

On subtracting  $C_{i-1,A} + A_{i,A} + A_{i+1,A} + t_i + t_{i+1} + A_{i,B} + A_{i+1,B}$  from each term, we have

$$\max \{-t_i - A_{i,B}, -t_{i+1} - A_{i+1,A}\} < \max \{-t_{i+1} - A_{i+1,B}, -t_i - A_{i,A}\}$$

$$\text{Or } \min \{t_i + A_{i,B}, t_{i+1} + A_{i+1,A}\} > \min \{t_{i+1} + A_{i+1,B}, t_i + A_{i,A}\}$$

$$\text{Or } \min \{A_{i,A} + t_i, A_{i+1,B} + t_{i+1}\} < \min \{A_{i+1,A} + t_{i+1}, A_{i,B} + t_i\}$$

Hence, the required result is verified.

## 4.2 Algorithm

The following algorithm gives the procedure to determine the optimal sequence of the jobs to minimize the idle time for the machines A and B when the completion time (waiting time + service time) of the jobs coming out of Phase I is the setup times for the machine A.

**Step 1.** Find the mean queue length on the lines of Gupta *et al.* [8] using the formula

$$L = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + \frac{\rho_3}{1-\rho_3} + \frac{\rho_4}{1-\rho_4} + \frac{\rho_5}{1-\rho_5}.$$

Here,

$$\rho_1 = \frac{(\lambda_1 + \lambda_2 p_{21})}{\mu_1 (1 - p_{12} p_{21})}, \rho_2 = \frac{(\lambda_2 + \lambda_1 p_{12})}{\mu_2 (1 - p_{12} p_{21})}, \rho_3 = \frac{\lambda'_1}{\mu'_1}, \rho_4 = \frac{\lambda'_2}{\mu'_2},$$

$$\rho_5 = \left[ \frac{\lambda'_1 + \lambda'_2}{\mu_3} + \frac{p_{13}(\lambda_1 + \lambda_2 p_{21}) + p_{23}(\lambda_2 + \lambda_1 p_{12})}{\mu_3 (1 - p_{12} p_{21})} \right];$$

where  $\lambda_i$  is the mean arrival rate,  $\mu_i$  is the mean service rate and  $p_{ij}$  are the probabilities.

**Step 2.** Find the average waiting time of the customers on the line of Little's [3] using relation  $E(w) = \frac{L}{\lambda}$ , where  $\lambda = \lambda_1 + \lambda_2 + \lambda'_1 + \lambda'_2$

**Step 3.** Find the completion time(C) of jobs/customers coming out of Phase I, .i.e. when processed through the network of queues  $Q_1, Q_2, Q_3, Q_4$  and  $Q_5$  by using the formula

$$C = E(W) + \frac{1}{\mu_1 p_{12} + \mu_1 p_{13} + \mu_2 p_{21} + \mu_2 p_{23} + \mu_3 + \mu'_1 + \mu'_2}.$$

**Step 4.** The completion time C of the customers / jobs through the network of queues  $Q_1, Q_2, Q_3, Q_4$  and  $Q_5$  will form the setup time for machine A. Define the two machines A and B with processing time  $A'_{i,A} = A_{i,A} + C$  and  $A_{i,B}$ .

**Step 5.** If  $t_i$  be the transportation time of  $i^{th}$  job from machine A to machine B. Introduce to fictitious machines  $G_i$  and  $H_i$  with processing times  $G_i = A'_{i,A} + t_i$  and  $H_i = A_{i,B} + t_i$

**Step 6.** Apply modified Johnson's procedure to find the optimal sequence(s) with minimum elapsed time using theorem 4.1.

**Step 7.** Prepare In-Out tables for the optimal sequence(s) obtained in step 6. The sequence  $S_k$  having minimum total elapsed time will be the optimal sequence for the given problem.

## 5 Numerical Illustration

Consider twelve customers / jobs are processed through the network of queues  $Q_1$ ,  $Q_2$ ,  $Q_3$ ,  $Q_4$  and  $Q_5$  with the servers  $S_1$ ,  $S_2$  and  $S_3$ . The server  $S_1$  consists of two biserial service servers  $S_{11}$  and  $S_{12}$ . The server  $S_2$  contains two parallel servers  $S_{21}$  and  $S_{22}$ . Server  $S_3$  is commonly linked in series with each of two servers  $S_1$  and  $S_2$ . The number of the customers, mean arrival rate, mean service rate and associated probabilities are given as in table 1.

**Table 1** The detail classification of the linkage model

| S. No. | No. of Customers | Mean Arrival Rate | Mean Service Rate | Probabilities  |
|--------|------------------|-------------------|-------------------|----------------|
| 1      | $n_1 = 2$        | $\lambda_1 = 6$   | $\mu_1 = 15$      | $p_{12} = 0.6$ |
| 2      | $n_2 = 3$        | $\lambda_2 = 4$   | $\mu_2 = 18$      | $p_{13} = 0.4$ |
| 3      | $n_3 = 4$        | $\lambda'_1 = 2$  | $\mu'_1 = 8$      | $p_{21} = 0.4$ |
| 4      | $n_4 = 2$        | $\lambda'_2 = 5$  | $\mu'_2 = 10$     | $p_{23} = 0.6$ |
| 5      | $n_5 = 11$       |                   | $\mu_3 = 20$      |                |

After getting service at Phase I jobs/customers are to be served at the machines  $M_1$  and  $M_2$  with processing time  $M_1$  and  $M_2$  respectively as given in table 2.

**Table 2** The machines  $M_1$  and  $M_2$  with processing times

| Jobs          | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---------------|---|---|---|---|---|---|---|---|---|----|----|
| $M_1(A_{i1})$ | 5 | 3 | 5 | 7 | 6 | 4 | 3 | 6 | 8 | 2  | 3  |
| $t_i$         | 2 | 1 | 3 | 2 | 1 | 2 | 1 | 2 | 3 | 1  | 2  |
| $M_2(A_{i2})$ | 7 | 4 | 6 | 8 | 8 | 5 | 7 | 4 | 4 | 5  | 4  |

The objective is to find an optimal sequence of the jobs / customers to minimize the makespan in this Queue-Scheduling linkage system by considering the first phase service into account.

**Solution:** We have

$$\rho_1 = \frac{\lambda_1 + \lambda_2 p_{21}}{(1 - p_{12} p_{21}) \mu_1} = 0.666667$$

$$\rho_2 = \frac{\lambda_2 + \lambda_1 p_{12}}{(1 - p_{12} p_{21}) \mu_2} = 0.55556$$

$$\rho_3 = \frac{\lambda'_1}{\mu'_1} = 0.25$$

$$\rho_4 = \frac{\lambda'_2}{\mu'_2} = 0.25$$

$$\rho_5 = \frac{\lambda'_1 + \lambda'_2}{\mu_3} + \frac{p_{13}(\lambda_1 + \lambda_2 p_{21}) + p_{23}(\lambda_2 + \lambda_1 p_{12})}{\mu_3(1 - p_{12}p_{21})} = 0.85$$

Mean Queue Length = Average number of Jobs / Customers =

$$L = \frac{\rho_1}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_2} + \frac{\rho_3}{1 - \rho_3} + \frac{\rho_4}{1 - \rho_4} + \frac{\rho_5}{1 - \rho_5} = 10.25 \text{ units.}$$

Average waiting time of the jobs / customers =  $E(w) = \frac{L}{\lambda} = 0.602020941$  units.

The total completion time of Jobs / Customers when processed through queue network in

$$\text{Phase I } C = E(W) + \frac{1}{\mu_1 p_{12} + \mu_1 p_{13} + \mu_2 p_{21} + \mu_2 p_{23} + \mu_3 + \mu'_1 + \mu'_2} = 0.617026 \text{ units.}$$

On taking the completion time  $C = 0.617026$  as the setup time, when jobs / customers came for processing with machine  $M_1$ . The new reduced problem with processing times  $A'_{i1} = A_{i1} + C$  and  $A_{i2}$  on machine  $M_1$  and  $M_2$  is as shown in table 3.

**Table 3** The processing times  $A'_{i1}$  and  $A_{i2}$  on machine  $M_1$  and  $M_2$  is

| Jobs | $A'_{i1}$ | $t_i$ | $A_{i2}$ |
|------|-----------|-------|----------|
| 1    | 5.617026  | 2     | 7        |
| 2    | 3.617026  | 1     | 4        |
| 3    | 5.617026  | 3     | 6        |
| 4    | 7.617026  | 2     | 8        |
| 5    | 6.617026  | 1     | 8        |
| 6    | 4.617026  | 2     | 5        |
| 7    | 3.617026  | 1     | 7        |
| 8    | 6.617026  | 2     | 4        |
| 9    | 8.617026  | 3     | 4        |
| 10   | 2.617026  | 1     | 5        |
| 11   | 3.617026  | 2     | 4        |

The two fictitious machines with processing times  $G_i$  and  $H_i$  by considering transportation time of the jobs from machine  $M_1$  to  $M_2$  are as follows

**Table 4** The processing times  $G_i$  and  $H_i$  on machine  $M_1$  and  $M_2$

| Jobs | $G_i$     | $H_i$ |
|------|-----------|-------|
| 1    | 7.617026  | 9     |
| 2    | 4.617026  | 5     |
| 3    | 8.617026  | 9     |
| 4    | 9.617026  | 10    |
| 5    | 7.617026  | 9     |
| 6    | 6.617026  | 7     |
| 7    | 4.617026  | 8     |
| 8    | 8.617026  | 6     |
| 9    | 11.617026 | 7     |
| 10   | 3.617026  | 6     |
| 11   | 5.617026  | 6     |

Using modified Johnson's algorithm as verified by the theorem proved in section, the optimal sequence of jobs processing is

$$S = 10 - 7 - 2 - 11 - 6 - 1 - 5 - 3 - 4 - 9 - 8.$$

The In-Out flow table for the sequence S is.

**Table 5** The In-Out flow table for the sequence S

| Jobs | Machine $M_1$     | Machine $M_2$     |
|------|-------------------|-------------------|
| 10   | 0.00000 – 2.61703 | 3.61703 – 8.61703 |
| 7    | 2.61703 – 6.23405 | 8.61703 – 16.617  |
| 2    | 6.23405 – 9.85108 | 16.617 – 21.617   |
| 11   | 9.85108 – 13.4681 | 21.617 – 27.617   |
| 6    | 13.4681 – 18.0851 | 27.617 – 34.617   |
| 1    | 18.0851 – 23.7022 | 34.617 – 43.617   |
| 5    | 23.7022 – 30.3192 | 43.617 – 52.617   |
| 3    | 30.3192 – 35.9362 | 52.617 – 61.617   |
| 4    | 35.9362 – 43.5532 | 61.617 - 71.617   |
| 9    | 43.5532 – 52.1703 | 71.617 – 78.617   |
| 8    | 52.1703 – 58.7873 | 78.617 – 84.617   |

Therefore, the total minimum elapsed time for sequence S of jobs is 84.617 units, average waiting time for the jobs is 0.60202941 units and mean queue length is 10.25 units.

## 6 Conclusions

The present paper establishes linkage between the queue network comprised of three servers  $S_1$ ,  $S_2$ ,  $S_3$  with a two stage flowshop scheduling system consisting machines  $M_1$  and  $M_2$ . The server  $S_1$  consists of two biserial service servers  $S_{11}$  and  $S_{12}$ . The server  $S_2$  contains two parallel servers  $S_{21}$ ,  $S_{22}$  and  $S_{23}$ . Server  $S_3$  is commonly linked in series with each of two servers  $S_1$  and  $S_2$  for completion of first phase service demanded either at a subsystem  $S_1$  or  $S_2$ . The objective of the model is to minimize the total elapsed time. A heuristic algorithm by considering the completion time of jobs in Phase I as setup time for the machine  $M_1$  in Phase II is discussed. The study may further be extended by generalizing the number of machines and by introducing various parameters like setup time, Breakdown Interval, Job Block Criteria etc.

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## Appendix

```
#include<iostream.h>
#include<stdio.h>
#include<conio.h>
#include<process.h>
#include<math.h>

int n[4],u[5],L[4];
int j[50],j1[50],m1;
float p[4];
float r[5];
float g[50],h[50],a[50],b[50],t1[50],g1[50],h1[50];
float a1,b1,a2,b2,a3,b3,a4,b4,b5,c1,c2,P,Q,V,W,M,z1,z2,z3,x;
float q1,q2,q3,z,f,c;
void main()
{
    clrscr();
    for( int i=1;i<=4;i++)
    {
        cout<<"Enter the number of customers and Mean Arrival Rate for Channel S"<<i<<":";
        cin>>n[i]>>L[i];
    }
    m1=n[1]+n[2]+n[3]+n[4];
    for(int d=1;d<=5;d++)
    {
        cout<<"\nEnter the Mean Service Rate for the Channel S"<<d<<":";
        cin>>u[d];
    }
    for(int k=1;k<=4;k++)
    {
```

```

        cout<<"\nEnter the value of probability p"<<k<<":";
        cin>>p[k];
    }
for(i=1;i<=m1;i++)
{
    j[i]=i;
    cout<<"\nEnter the processing time  of "<<i<<" job for machine A : ";
    cin>>a[i];
    cout<<"\nEnter the transportation time of "<<i<<" job form machine A to machine B:
",
    cin>>t1[i];
    cout<<"\nEnter the processing time  of "<<i<<" job for machine B : ";
    cin>>b[i];
}
a1=L[1]+L[2]*p[3];
b1=(1-p[1]*p[3])*u[1];
r[1]=a1/b1;
a2=L[2]+L[1]*p[1];
b2=(1-p[1]*p[3])*u[2];
r[2]=a2/b2;
a3=L[3];a4=L[4];b3=u[3];b4=u[4];b5=u[5];
r[3]=a3/b3;
r[4]=a4/b4;
c2=(1-p[1]*p[3])*b5;
z1=(a3+a4)/b5;
z2= a1*p[2]/c2;
z3= a2*p[4]/c2;
r[5]=z1+z2+z3;
M=L[1]+L[2]+L[3]+L[4];
for(i=1;i<=5;i++)
{
    cout<<"r["<<i<<"]\t\t"<<r[i]<<"\n";
}
for(i=1;i<=5;i++)
{
    if(r[i]>1)
    {
        cout<<"Steady state condition does not holds good for"<<r[i]<<"...\nExiting";
        getch();
        exit(0);
    }
}
Q =(r[1]/(1-r[1]))+(r[2]/(1-r[2]))+(r[3]/(1-r[3]))+(r[4]/(1-r[4]))+(r[5]/(1-r[5]));
cout<<"\nThe mean queue length is : "<<Q<<"\n";
W=Q/M;
cout<<"\nAverage waiting time for the customer is:"<<W<<"\n";
z=u[1]*p[1]+u[1]*p[2]+u[2]*p[3]+u[2]*p[4]+u[3]+u[4]+u[5];
f=1/z;

```

```

c= W+f;
cout<<"\n\nTotal completion time of Jobs / Customers through Queue Network in
Phase 1 : "<<c;
for(i=1;i<=m1;i++)
{
    g1[i]=a[i]+c;
    h1[i]=b[i];
}
for(i=1;i<=m1;i++)
{
    g[i]=g1[i]+t1[i];
    h[i]=h1[i]+t1[i];
}
for(i=1;i<=m1;i++)
{
    cout<<"\n\n"<<j[i]<<"\t"<<g[i]<<"\t\t"<<t1[i]<<"\t"<<h[i];
    cout<<endl;
}
float mingh[16];
char ch[16];
for(i=1;i<=m1;i++)
{
    if(g[i]<h[i])
    {
        mingh[i]=g[i];
        ch[i]='g';
    }
    else
    {
        mingh[i]=h[i];
        ch[i]='h';
    }
}
for(i=1;i<=m1;i++)
{
    cout<<endl<<mingh[i]<<"\t"<<ch[i];
}
for(i=1;i<=m1;i++)
{
    for(int k=1;k<=m1;k++)

        if(mingh[i]<mingh[k])
        {
            float temp=mingh[i]; int temp1=j[i]; char
d=ch[i];

            mingh[i]=mingh[k]; j[i]=j[k]; ch[i]=ch[k];
            mingh[k]=temp; j[k]=temp1; ch[k]=d;
        }
}

```

```

    }
    for(i=1;i<=m1;i++)
    {
        cout<<endl<<endl<<j[i]<<"\t"<<mingh[i]<<"\t"<<ch[i]<<"\n";
    }

// calculate scheduling
float sbeta[16];

int t=1,s=0;
for(i=1;i<=m1;i++)
{
    if(ch[i]=='h')
    {
        sbeta[(m1-s)]=j[i];
        s++;
    }
else if(ch[i]=='g')
    {
        sbeta[t]=j[i];
        t++;
    }
}

int arr1[16], m=1;
    cout<<endl<<endl<<"Job Scheduling:"<<"\t";
for(i=1;i<=m1;i++)
{
    cout<<sbeta[i]<<" ";
    arr1[m]=sbeta[i];
    m++;
}

//calculating total computation sequence
float macha[50], machb[50],maxv1[50];
float time=0.0;
macha[1]=time+g1[arr1[1]];
for(i=2;i<=m1;i++)
{macha[i]=macha[i-1]+g1[arr1[i]];}
machb[1]=macha[1]+h1[arr1[1]]+t1[arr1[1]];
for(i=2;i<=m1;i++)
{
    if((machb[i-1])>(macha[i]))
        maxv1[i]=machb[i-1];
else
    maxv1[i]=macha[i];machb[i]=maxv1[i]+h1[arr1[i]]+t1[arr1[i]];
}

//displaying solution
cout<<"\n\n\n\n\n\t\t\t #####THE SOLUTION##### ";
cout<<"\n\n\t*****";

```

```

cout<<"\n\n\n\t Optimal Sequence is : ";
for(i=1;i<=m1;i++)
    {cout<<" "<<arr1[i];}
cout<<endl<<endl<<"In-Out Table is:"<<endl<<endl;
cout<<"Jobs"<<"\t"<<"Machine M1"<<"\t"<<"\t"<<"Machine M2" <<"\t"<<endl;
cout<<arr1[1]<<"\t"<<time<<"--"<<macha[1]<<"\t"<<"\t"<<t1[arr1[1]]+macha[1]<<"--"
"<<machb[1]<<"\t"<<endl;
for(i=2;i<=m1;i++)
    {
cout<<arr1[i]<<"\t"<<macha[i-1]<<"--"<<macha[i]<<"\t"<<"\t"<<maxv1[i]<<"--"
"<<machb[i]<<"\t"<<endl;
    }
cout<<"\n\n\nTotal Elapsed Time (T) = "<<machb[m1];
cout<<"\n\n\t*****";
getch();
}

```