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Shortest Path Problem with Gamma Probability Distribution Arc Length

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Abstract We propose a dynamic program to find the shortest path in a network having gamma probability distributions as arc lengths. Two operators of sum and comparison need to be adapted for the proposed dynamic program. Convolution approach is used to sum two gamma probability distributions being employed in the dynamic program.

Keywords Shortest Path, Dynamic Program, Convolution, Gamma Distribution.

1 Introduction

Deterministic, time-dependent shortest path problems have been widely studied for the case of determining a single shortest path. If the arc lengths are constant, there are several efficient algorithms developed by [1,2,3]. Cook and Halsey [4] extended Bellman's principle of optimality for dynamic programming (1958) to this case and Dreyfus (1969) suggested the use of Dijkstra's algorithm (1959) for determining time-dependent shortest paths. Halpern [5] noted the limitations of the approach of Dreyfus (1969). It should be noted that the standard shortest path algorithms also have been found to be applicable to compute shortest paths in time-dependent but not stochastic networks [6, 7, 8, 9].

Kaufman *et al.,* [10] subsequently studied the assumptions under which the existing time-dependent shortest path problems algorithms would work, and showed that if the link-delays follow the first-in-first-out (FIFO) rule or consistency assumption, then one could use an expanded static network to obtain optimal paths. Malandraki [11] analyzed the time-dependent shortest path problem and extended Halpern's result for the special case of differentiable link delay functions and showed that the consistency assumption would be satisfied by verifying that the first derivative of the link delay function did not exceed negative unity.

Ziliaskopoulos and Mahmassani [12] noted that turning movements of vehicles in congested urban networks contribute significantly to the travel time. The authors prescribed an efficient label-correcting procedure that uses an extended forward-star structure to represent the network including intersection movements and movement prohibitions. Chen and Tang [13] analyzed a shortest path problem on a mixed-schedule network, subject to side constraints. Haquari and Dejax [14] analyzed a similar problem, considering time-varying

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costs and knapsack-like constraints. However, due to failure, maintenance or other reasons, different kinds of uncertainties are frequently encountered in practice, and must be taken into account. For example, the lengths of the arcs are assumed to represent transportation time or cost rather than the geographical distances, as time or cost fluctuate with traffic or weather conditions, payload and so on, it is not practical to consider each arc as a deterministic value. In these cases, probability theory has been used to attack randomness, and many researchers have done lots of work on stochastic shortest path problem.

When arc lengths are random variables, the problem will become more difficult. Frank [15] computed the probability that the time of the shortest path of the network is smaller than a specific value where link travel times are random variables but not time dependent. Loui [16], Mirchandani and Soroush [17], and Murthy and Sarkar [18] showed that for identifying the expected shortest path if the random link travel times are replaced by their expected values, then the problem simply reduces to a deterministic shortest path problem and standard shortest path algorithms still can be used to find the expected shortest paths in a network. Wijerante et al. [19] presented a method to find the set of non-dominated paths from the source node to the sink node, in which each arc includes several criteria that some of them might be stochastic. Carraway et al. [20] applied the method of generalized dynamic programming to find the optimal path of a bicriteria network. Hall [21] studies *for the* first time the time-dependent version of the shortest path problem. He demonstrated that the standard shortest path algorithm may fail to find the expected shortest path in these networks. Hall proposed an optimal dynamic programming based algorithm to find the shortest paths and this algorithm was demonstrated on a small transit network example. He showed that the optimal ‘‘route choice’’ is not a simple path but an adaptive decision rule. The best route from any given node to the final destination depends on the arrival time at that node. The paper only considers the case where link travel times are modeled as discrete-time stochastic processes.

Fu [22] studied the expected shortest paths in dynamic and stochastic networks in a traffic network where the link travel times are modeled as a continuous-time stochastic process. He showed that the replacement of the probability distribution for link delays by their expected values would yield sub-optimal results and prescribed a dynamic programming algorithm to solve the problem using conditional probability theory. Kaufman and Smith [10] subsequently showed that the time-space network formulation and expected link delays could be used to solve the problem if the consistency assumption is satisfied. Fan et al. [23] minimize expected travel time from any origin to a specific destination in a congestible network with correlated link costs. Bertsimas and Van Ryzin [24] introduced and analyzed a model for stochastic and dynamic vehicle routing, in which a single, uncapacitated vehicle traveling at a constant velocity in a Euclidean region must serve demands whose time of arrival, location and on-site service are stochastic. [25] extended this analysis and considered the problem of m identical vehicles with unlimited capacity.

Miller et al. [26] prescribed an efficient label correcting algorithm to obtain Pareto optimal paths by discretizing the probability distribution of the link delays. Psaraftis and Tsitsiklis [27] examined shortest path problems, in which arc costs are the known functions of certain environmental variables at network nodes, and each of these variables evolves according to an independent Markov process. The vehicle can wait at a node in anticipation of more favorable arc costs. They showed that the optimal policy essentially classifies the state of the environmental variable at a node into two categories: green states for which the optimal action is to immediately traverse the arc, and red states for which the optimal action is to wait. Then they extended these concepts for the entire network by developing a dynamic

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programming, which solves the corresponding problem. Ji [28] studied the shortest path problem with stochastic arc length. According to different decision criteria and presented three types of models, In order to solve these models, a hybrid intelligent algorithm integrating stochastic simulation and genetic algorithm has been developed.

2 Problem definition

Consider a network as shown in Figure 1 consisting of a finite set of nodes and arcs of the directed acyclic network. We assume that the admissible paths are always continuous and always move toward the right, and the length of each arc is a gamma random variable. We want to find the shortest path from the source node 1 to the sink node N using the backward dynamic programming approach.

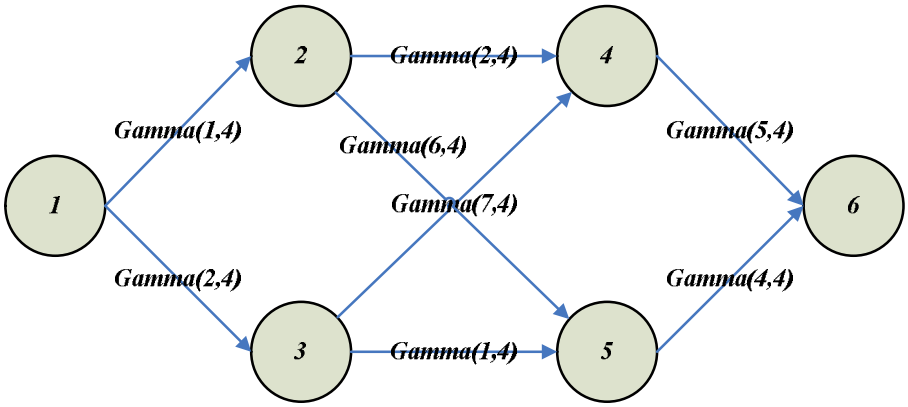

Fig. 1 A network with same rate parameter gamma distribution

The optimal value function S_i can be defined by

S_i = the distribution of the shortest path from node i to node N.

Then the recurrence relation can be stated as

$$S_i = \min_{j>i}[d_{ij} + S_j] \qquad \text{For} \quad i = N\text{-}1,\ldots 1 \tag{1}$$

and the boundary condition is $S_N = 0$.

In this paper we use convolution to find distribution of sum of two gamma distributions in each stage. And for comparison in each stage we find the probability that a random variable with first distribution become smaller than another random variable with second distribution.

In order to show the operation in each stage, we first represent the convolution and comparison between two gamma distributions with same rate parameter, and then we show the convolution and comparison between two gamma distributions with different rate parameter.

Definition 1. Let X and Y be two continuous random variables with density functions $f(x)$ and $g(y)$, respectively. Assume that both $f(x)$ and $g(y)$ are defined for all real numbers. Then the *convolution* $f * g$ of f and g is the function given by

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$$(f * g)(z) = \int_{-\infty}^{+\infty} f(x)\, g(z-x)\, dx$$
$$= \int_{-\infty}^{+\infty} f(z-y)\, g(y)\, dy$$

Theorem 1. Let X and Y be two independent random variables with density functions $f_X(x)$ and $f_Y(y)$ defined for all x. Then the sum $Z = X + Y$ is a random variable with density function $f_Z(z)$, where f_Z is the convolution of f_X and f_Y.

$$f_z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(x, z-x)\, dx \qquad (2)$$
$$= \int_{-\infty}^{+\infty} f_{X,Y}(z-y, y)\, dy$$

Proof. as we knew the joint density function of independent variables is equal to the products of their density functions therefore to find density function of Z = X + Y we apply cumulative distribution function technique.

$$P(Z \le z) = P(X+Y \le z) = \int_{-\infty}^{+\infty} P(X+Y \le z | X = x)\, f_X(x)$$
$$= \int_{-\infty}^{+\infty} P(x+y \le z)\, f_X(x)\, dx = \int_{-\infty}^{+\infty} F_Y(z-x)\, f_X(x)\, dx$$

Now, we set partial derivative to obtain the summation density function

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{d}{dz}\left[\int_{-\infty}^{+\infty} F_Y(z-x)\, f_X(x) dx\right]$$
$$= \int_{-\infty}^{+\infty} \frac{dF_Y(z-x)}{dz}\, f_X(x) dx = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x) dx$$

2.1 Sum of two independent gamma random variables with same rate parameter

Suppose that we have two random variables X and Y with a *gamma* density function with parameter $\lambda > 0$ *and* $\alpha > 0$. We represent the density function of Z = X + Y as follows

$$f_X(x) = \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{-\lambda x} \qquad x \ge 0,$$
$$f_Y(y) = \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} y^{\alpha_2 - 1} e^{-\lambda y} \qquad y \ge 0.$$

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$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x)\, dx$$
$$= \int_0^z \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)}(z-x)^{\alpha_2-1}e^{-\lambda(z-x)}\ \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)}x^{\alpha_1-1}e^{-\lambda x}\, dx$$
$$= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)}z^{\alpha_1+\alpha_2-1}e^{-\lambda z} \tag{3}$$

As a result, if $X_1, X_2,\ldots, X_n$ are independent gamma random variables with (α_1,λ), $(\alpha_2,\lambda),\ldots,$ (α_n,λ) then $Y = \sum_{i=1}^{n} X_i$ follows gamma distribution with λ and $\alpha = \sum_{i=1}^{n}\alpha_i$.

$$f_Y(y) = \frac{\lambda^{\alpha_1+\cdots+\alpha_n}}{\Gamma(\alpha_1+\cdots+\alpha_n)}\, x^{\alpha_1+\cdots+\alpha_n-1}\, e^{-\lambda x} \tag{4}$$

2.2 Sum of independent gamma random variables with different rate parameter

Suppose that we have two random variables $X_1, X_2,\ldots, X_n$ with a gamma density function with parameter (α_1,λ_1), $(\alpha_2,\lambda_2),\ldots,$ (α_n,λ_n). We represent the density function of $Y = \sum_{i=1}^{n} X_i$ as follows

$$f_Y(y) = C\, y^{\alpha_1+\cdots+\alpha_n-1}\int_0^1\ldots\int_0^1 e^{-yC_{\lambda_1\ldots,\lambda_n(u_1,\ldots,u_{n-1})}}\, B_{\alpha_1,\ldots,\alpha_n}(u_1,\ldots,u_{n-1})du_1\ldots du_{n-1} \tag{5}$$

For all x>0 and $f_Y(y) = 0$ for all $x \le 0$, where

$$C = \frac{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\ldots\lambda_n^{\alpha_n}}{\Gamma(\alpha_1+\cdots+\alpha_n)} \tag{6}$$

and

$$C_{\lambda_1\ldots,\lambda_n(u_1,\ldots,u_{n-1})} := \lambda_1\ \prod_{j=1}^{n-1}u_j + \sum_{i=2}^{n-1}\lambda_i\left(1-u_j\right)\prod_{j=i}^{n-1}u_j + \lambda_n\left(1-u_{n-1}\right) \tag{7}$$

and

$$B_{\alpha_1,\ldots,\alpha_n}(u_1,\ldots,u_{n-1}) := \frac{1}{B_n(\alpha_1,\ldots,\alpha_n)}\prod_{j=1}^{n-1}u_j^{\alpha_1+\cdots+\alpha_n-1}(1-u_j)^{\alpha_{j+1}-1} \tag{8}$$

For all $u_1,\ldots,u_{n-1} \in [0,1]$.

See (Akkuchi, 2005).

Now we illustrate the method that we use to find minimum between two gamma random variables. In order to find the minimum random variable we compute the probability that the

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first random variable X_1 with gamma density function with (α_1,λ_1) became smaller than the second random variable X_2 with gamma density function with (α_2,λ_2).

$$P(X_1 < X_2) = \int_0^{\infty} P(X_1 < X_2 | X_1 = x_1)\,.\, f_{x_1}(x_1)\, dx_1$$
$$= \int_0^{\infty}\int_{x_1}^{\infty} \frac{{\lambda_2}^{\alpha_2}}{\Gamma(\alpha_2)} {x_2}^{\alpha_2-1} e^{-\lambda_2 x_2} . \frac{{\lambda_1}^{\alpha_1}}{\Gamma(\alpha_1)} {x_1}^{\alpha_1-1} e^{-\lambda_1 x_1}\, d\, x_2\, dx_1$$

3 A numerical example

Consider the network depicted in Figure 1. We want to obtain the shortest path from node 1 to node 6 where arcs have gamma distribution with same rate parameter.
Boundary condition is $S_6 = 0$.
Using the recurrence relation (1) we have

$$S_5 = Gamma(4,4), S_4 = Gamma(5,4)$$

For each arc that doesn't exist in network we replace infinity for d_{ij} in relation (1).

$$S_3 = min\begin{bmatrix} Gamma(7,4) + S_4 \\ Gamma(1,4) + S_5 \end{bmatrix} = min\begin{bmatrix} Gamma(7,4) + Gamma(5,4) \\ Gamma(1,4) + Gamma(4,4) \end{bmatrix}$$
$$= min\begin{bmatrix} Gamma(12,4) \\ Gamma(5,4) \end{bmatrix}$$

We find the minimum value between two density function as follows

$$P(X_1 < X_2) = \int_0^{\infty} P(X_1 < X_2 | X_1 = x_1)\,.\, f_{x_1}(x_1)\, dx_1$$
$$= \int_0^{\infty}\int_{x_1}^{\infty} \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} {x_2}^{\alpha_2-1} e^{-\lambda x_2} . \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} {x_1}^{\alpha_1-1} e^{-\lambda x_1}\, d\, x_2\, dx_1 =$$
$$= \int_0^{\infty}\int_{x_1}^{\infty} \frac{4^5}{4!} {x_2}^4 e^{-4x_2} . \frac{4^{12}}{11!} {x_1}^{11} e^{-4x_1}\, d\, x_2\, dx_1 = 0.038$$

With probability 0.038 the first density function is smaller than the second, so we choose the second density function as minimum.

$$S_3 = Gamma(5,4)$$

We illustrate the operation of node 2 as follows

$$S_2 = min\begin{bmatrix} Gamma(2,4) + S_4 \\ Gamma(6,4) + S_5 \end{bmatrix} = min\begin{bmatrix} Gamma(2,4) + Gamma(5,4) \\ Gamma(6,4) + Gamma(4,4) \end{bmatrix}$$
$$= min\begin{bmatrix} Gamma(7,4) \\ Gamma(10,4) \end{bmatrix}$$

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and too finding minimum value we have

$$P(X_1 < X_2) = \int_0^{\infty} P(X_1 < X_2 | X_1 = x_1)\,.\, f_{x_1}(x_1)\, dx_1$$
$$= \int_0^{\infty}\int_{x_1}^{\infty} \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} {x_2}^{\alpha_2-1} e^{-\lambda x_2} . \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} {x_1}^{\alpha_1-1} e^{-\lambda x_1}\, d\, x_2\, dx_1 =$$
$$= \int_0^{\infty}\int_{x_1}^{\infty} \frac{4^{10}}{9!} {x_2}^{9} e^{-4x_2} . \frac{4^{7}}{6!} {x_1}^{6} e^{-4x_1}\, d\, x_2\, dx_1 = \frac{50643}{65536} = 0.772$$

with probability 0.772 the first density function is smaller than the second, so we choose the first density function as minimum.

$$S_2 = Gamma(7,4)$$

Now we do operations for S_1 to find the shortest path in network

$$S_1 = min\begin{bmatrix} Gamma(1,4) + S_2 \\ Gamma(2,4) + S_3 \end{bmatrix} = min\begin{bmatrix} Gamma(1,4) + Gamma(7,4) \\ Gamma(2,4) + Gamma(5,4) \end{bmatrix}$$
$$= min\begin{bmatrix} Gamma(8,4) \\ Gamma(7,4) \end{bmatrix}$$

$$P(X_1 < X_2) = \int_0^{\infty}\int_{x_1}^{\infty} \frac{4^{7}}{6!} {x_2}^{6} e^{-4x_2} . \frac{4^{8}}{7!} {x_1}^{7} e^{-4x_1}\, d\, x_2\, dx_1 = \frac{1619}{4096} = 0.3952$$

$$min\begin{bmatrix} Gamma(8,4) \\ Gamma(7,4) \end{bmatrix} = Gamma(7,4)$$

With probability 0.3952 the first density function is smaller than the second, so we choose the second density function as minimum. Now the shortest path in network with probability 0.6048 is 1-3-5-6.
Now we explain operations to find the shortest path in network with gamma arcs in general case.

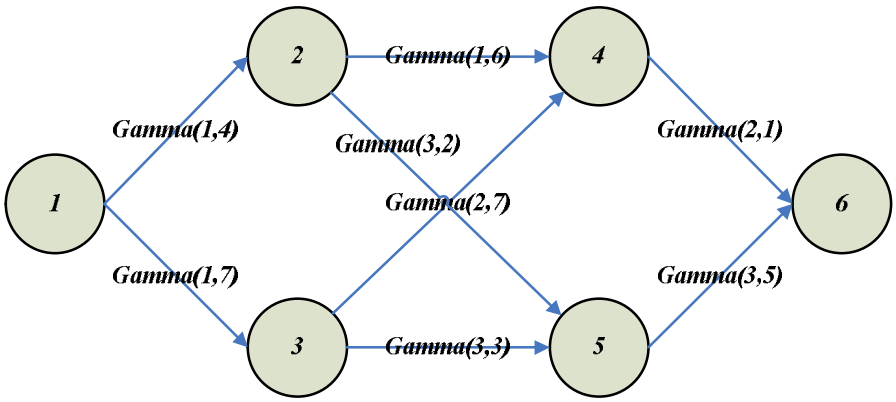

Fig. 2 A network with gamma distribution arc length

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Boundary condition is $S_6 = 0$
Using the recurrence relation (1) we have

$$S_5 = Gamma(3,5),\ S_4 = Gamma(2,1)$$

For each arc that doesn't exist in network we replace infinity for d_{ij} in relation (1).

$$S_3 = min\begin{bmatrix} Gamma(2,7) + S_4 \\ Gamma(3,3) + S_5 \end{bmatrix} = min\begin{bmatrix} Gamma(2,7) + Gamma(2,1) \\ Gamma(3,3) + Gamma(3,5) \end{bmatrix}$$

From formula (5) we have

$$X_1 = Gamma(2,7), X_2 = Gamma(2,1)$$

$$S_r = X_1 + X_2$$

$$f_Y(x) = 7^2\, e^{-7x}\left(\frac{(-1)^{2-1}}{0!}\, x^0 \binom{2+1-1}{1} \frac{1}{(-6)^3} + \frac{(-1)^0}{1!} x^1 \binom{2+0-1}{0} \frac{1}{(-6)^2}\right)$$

$$+1^2\, e^{-x}\left(\frac{(-1)^1}{0!}\, x^0 \binom{2+1-1}{1} \frac{7^2}{(6)^3} + \frac{(-1)^0}{1!} x^1 \binom{2+0-1}{0} \frac{7^2}{(6)^2}\right)$$

$$= 49e^{-7x}\left(\frac{2}{6^3} + \frac{x}{36}\right) + e^{-x}\left(\frac{-98}{6^3} + \frac{49x}{36}\right)$$

Convolution of gamma(3,3) and gamma(3,5) is as follows

$$f_X(x) = \frac{3^3}{(3-1)!} x^2 e^{-3x} \qquad x \geq 0\,,$$

$$f_Y(y) = \frac{5^3}{(3-1)!} y^2 e^{-5y} \qquad y \geq 0\,,$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x)\, dx$$

$$= \int_0^z \frac{{\lambda_2}^{\alpha_2}}{\Gamma(\alpha_2)} (z-x)^{\alpha_2 - 1} e^{-\lambda_2 (z-x)}\, \frac{{\lambda_1}^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1 - 1} e^{-\lambda_1 x}\, dx$$

$$= \int_0^z \frac{5^3}{(3-1)!} (z-x)^2 e^{-5(z-x)}\, \frac{3^3}{(3-1)!} x^2 e^{-3x}\, dx$$

$$= \frac{-10125}{16} e^{-5z} z - \frac{10125}{16} e^{-5z} - \frac{3375}{16} e^{-5z} z^2 - \frac{10125}{16} e^{-3z} z + \frac{10125}{16} e^{-3z} + \frac{3375}{16} e^{-3z} z^2 \qquad , z \geq 0$$

We find the minimum value between two density function as follows

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$$P(X_1 < X_2) = \int_0^{\infty} \int_{x_1}^{\infty} f_{x_2}(x_2) \,.\, f_{x_1}(x_1)\, dx_2 dx_1 \;= \int_0^{\infty} f_{x_1}(x_1) \left(\int_{x_1}^{\infty} f_{x_2}(x_2) dx_2 \right) dx_1$$

$$= \int_0^{\infty} \left(49e^{-7x_1} \left(\frac{2}{6^3} + \frac{x_1}{36} \right) + e^{-x_1} \left(\frac{-98}{6^3} + \frac{49x_1}{36} \right) \right) \left(\int_{x_1}^{\infty} \frac{-10125}{16} e^{-5x_2} x_2 - \frac{10125}{16} e^{-5x_2} \right.$$

$$- \frac{3375}{16} e^{-5x_2} {x_2}^2 - \frac{10125}{16} e^{-3x_2} x_2 + \frac{10125}{16} e^{-3x_2}$$

$$\left. + \frac{3375}{16} e^{-3x_2} {x_2}^2 \, dx_2 \right) \, dx_1 \;= \frac{200557}{552960} = 0.3627$$

With probability 0.3627 the first density function is smaller than the second, so we choose the second density function as minimum.

$$S_3 = min \begin{bmatrix} Gamma(2,7) + Gamma(2,1) \\ Gamma(3,3) + Gamma(3,5) \end{bmatrix} = \; Gamma(3,3) + Gamma(3,5)$$

We explain operations of node 2 as follows

$$S_2 = min \begin{bmatrix} Gamma(1,6) + S_4 \\ Gamma(3,2) + S_5 \end{bmatrix} = min \begin{bmatrix} Gamma(1,6) + Gamma(2,1) \\ Gamma(3,2) + Gamma(3,5) \end{bmatrix}$$

Convolution of gamma (1,6) and gamma(2,1) is as follows

$$f_X\,(x) = \frac{6^1}{0!} x^0 e^{-6x} \qquad x \geq 0\,,$$

$$f_Y\,(y) = \frac{1^2}{1!} y^1 e^{-y} \qquad y \geq 0\,,$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x)\, dx$$

$$= \int_0^z 6e^{-6x} \; (z-x)^1 e^{-(z-x)}\, dx \;= \frac{6}{5} e^{-z} z - \frac{6}{25} e^{-z} + \frac{6}{25} e^{-6z} \qquad , z \geq 0$$

and the convolution of gamma(3,2) and gamma(3,5) is as follows

$$f_X\,(x) = \frac{2^3}{2!} x^2 e^{-2x} \qquad x \geq 0\,,$$

$$f_Y\,(y) = \frac{5^3}{2!} y^2 e^{-5y} \qquad y \geq 0\,,$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z-x)\, f_X(x)\, dx$$

$$= \int_0^z \frac{2^3}{2!} x^2 e^{-2x} \cdot \frac{5^3}{2!} (z-x)^2 e^{-5(z-x)}\, dx$$

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$$= \frac{-500}{27} e^{-5z} z^2 - \frac{1000}{27} e^{-5z} z - \frac{2000}{81} e^{-5z} + \frac{500}{27} e^{-2z} z^2 - \frac{1000}{27} e^{-2z} z + \frac{2000}{81} e^{-2z} \quad , z \geq 0$$

We find the minimum value between two density function as follows

$$P(X_1 < X_2) = \int_0^{\infty} \int_{x_1}^{\infty} f_{x_2}(x_2) \, . \, f_{x_1}(x_1) \, dx_2 dx_1 = \int_0^{\infty} f_{x_1}(x_1) \left(\int_{x_1}^{\infty} f_{x_2}(x_2) dx_2 \right) dx_1$$

$$= \int_0^{\infty} \left(\frac{-500}{27} e^{-5x_1} {x_1}^2 - \frac{1000}{27} e^{-5x_1} x_1 - \frac{2000}{81} e^{-5x_1} + \frac{500}{27} e^{-2x_1} {x_1}^2 - \frac{1000}{27} e^{-2x_1} x_1 + \frac{2000}{81} e^{-2x_1} \right) \left(\int_{x_1}^{\infty} \frac{6}{5} e^{-x_2} x_2 - \frac{6}{25} e^{-x_2} + \frac{6}{25} e^{-6x_2} dx_2 \right) dx_1 = \frac{10902337}{20699712}$$

$$= 0.5267$$

With probability 0.5267 the first density function is smaller than the second, so we choose the first density function as minimum.

$$S_2 = min \begin{bmatrix} Gamma(1,6) + Gamma(2,1) \\ Gamma(3,2) + Gamma(3,5) \end{bmatrix} = Gamma(1,6) + Gamma(2,1)$$

Now we do operations for S_1 to find the shortest path in network

$$S_1 = min \begin{bmatrix} Gamma(1,4) + S_2 \\ Gamma(1,7) + S_3 \end{bmatrix}$$

$$= min \begin{bmatrix} Gamma(1,4) + Gamma(1,6) + Gamma(2,1) \\ Gamma(1,7) + Gamma(3,3) + Gamma(3,5) \end{bmatrix}$$

To obtain convolution of gamma (1,4), gamma (1,6) and gamma (2,1) we convolute gamma (1,4) with density function that obtained from convoluting gamma (1,6) and gamma (2,1).

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z - x) \, f_X(x) \, dx$$

$$= \int_0^z 4e^{-4(z-x)} \left(\frac{6}{5} e^{-x} x - \frac{6}{25} e^{-x} + \frac{6}{25} e^{-6x} \right) dx$$

$$= \frac{4}{75} (25e^{2z} + 30ze^{5z} - 16e^{5z} - 9)e^{-6z}$$

To obtain convolution of gamma (1,7), gamma (3,3) and gamma (3,5) we do same as above.

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(z - x) \, f_X(x) \, dx$$

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$$= \int_0^z 7e^{-7(z-x)} \left(-\frac{10125}{16} e^{-5x} x - \frac{10125}{16} e^{-5x} - \frac{3375}{16} e^{-5x} x^2 - \frac{10125}{16} e^{-3x} x \right.$$
$$\left. + \frac{10125}{16} e^{-3x} + \frac{3375}{16} e^{-3x} x^2 \right) dx$$
$$= \frac{-23625}{512} (-1 + 32ze^{2z} + 32e^{2z} + 16e^{2z} z^2 + 28ze^{4z} - 31e^{4z} - 8e^{4z} z^2) e^{-7z}$$

We find the minimum value between two density function as follows

$$P(X_1 < X_2) = \int_0^\infty \int_{x_1}^\infty f_{x_2}(x_2) \, . \, f_{x_1}(x_1) \, dx_2 dx_1 = \int_0^\infty f_{x_1}(x_1) \left(\int_{x_1}^\infty f_{x_2}(x_2) dx_2 \right) dx_1$$
$$= \int_0^\infty \left(\frac{4}{75} (25e^{2x_1} + 30x_1 e^{5x_1} - 16e^{5x_1} \right.$$
$$\left. - 9) e^{-6x_1} \right) \left(\int_{x_1}^\infty \frac{-23625}{512} (-1 + 32x_2 e^{2x_2} + 32e^{2x_2} + 16e^{2x_2} {x_2}^2 \right.$$
$$\left. + 28x_2 e^{4x_2} - 31e^{4x_2} - 8x_2) e^{-7x_2} dx_2 \right) dx_1 = \frac{103463451397}{281295286272} = 0.3678$$

Considering the probability of 0.3678, the first density function is smaller than the second one, so we choose the second density function as minimum. Now the shortest path in the network is 1-3-5-6 with probability of 0.6322.

4 Conclusions

This paper proposed a dynamic program for determining the shortest path in a gamma probability distribution arc length network. Since the definite values of the dynamic program were turned into gamma random variables, two modifications were performed on sum and comparison operators. The convolution technique was employed for summing two gamma probability distributions. The numerical example via a six node network showed the performance of the proposed methodology for the shortest path. The examples were reported in two cases with the same rate parameter and different one, respectively.

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