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A One-Model Approach for Computation of Congestion with Productions Trade-Offs and Weight Restrictions

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Abstract This paper focuses on the determine congestion with production trade-offs or, alternatively, through weights restrictions in data envelopment analysis (DEA). For this purpose, the researchers review a one-model approach to evaluate congestion Cooper *et al.,* [1], and then briefly review the calculation of efficient targets with production trade-offs in Podinovski's procedure [2]. Therefore, the method used in this study is a hybrid of the two above-said procedures: calculation of the congestion via weight restrictions, which is then supported by an experimental example by Podinovski [3].

Keywords Data Envelopment Analysis (DEA), Congestion, Trade-Offs, Weight Restrictions.

1 Introduction

Charnes *et al.,* (1978) introduced Data Envelopment Analysis (DEA) as a "data-oriented" approach for evaluating the performance of a set of peer entities named Decision-Making Units (DMUs), which convert multiple inputs into multiple outputs [4]. DMU may be defined flexibly and in a generic way. During the recent years DEA has been vastly used in a variety of applications in order to evaluate the performances various kinds of entities engaged in different activities in various contexts throughout the world. Various DMUs have been used in DEA applications to evaluate the performance of entities, such as hospitals, US Air Force wings, universities, cities, courts, business firms, and others, as well as the performance of countries, regions, and so on.

Congestion may be used in a vast variety of disciplines ranging from medical sciences to traffic engineering. The term can be used in everyday life as well. Congestion refers to an economic state where inputs are overly invested; therefore, congestion takes place when reducing some inputs can lead to an increase in outputs. For the first time, Färe and Grosskopf (1983) introduced an application form in order to quantitatively analyze congestion [5]. Later,

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Färe et al. (1985) discussed data envelopment analysis (or activity analysis Shephard [6] related models and methods (called FGL approach) for production efficiency evaluation [7]. Cooper et al. (1996) put forward an alternative DEA approach (CTT approach) to examine the congestion [8]. Cooper et al. (2001c), dealt FGL approach and CTT approach in more detail and compared them through numerical examples used to show different the advantages of both approaches [9]. Brockett et al. (1998) used the CTT approach to study the tradeoffs between employment and output which could be used to increase employment or increase output in the Chinese industry production, Cooper et al. (2000b) [11,10]. The management of congestion in the Chinese industry was further studied by Cooper et al. (2001a), and they illustrated how elimination of managerial inefficiencies could result in an increase in output without employment in textiles and automobile industries being reduced [12]. Cooper et al. (2002) introduced a new approach, which integrates these two phases in CTT approach into a single model approach [1].

The fact that DEA models are free to select weights in real-world usage may result in weighting schemes that are not compatible with prior knowledge or are contradictory to accepted views and beliefs or seem unrealistic in terms of management issues. In order to overcome these difficulties, one can incorporate information about the relative importance of the inputs and outputs involved or some managerial preferences into the basic DEA model, if, of course, these are available. Now the obtained efficiency scores can evaluate both the technical inefficiency that results from the production possibilities not totally exploited and the inefficiency that can be a result of either the managerial goals not fulfilled or the moving away from the specified value system of the inputs and outputs.

In DEA, the production possibility set (PPS) is estimated from the data of all the DMUs together with some general axioms assumed for the underlying technology (see Banker et al. (1984) [13]). If we have information about realistic technological trade-offs, then we can use it in order to increase the set of possible input–output combinations beyond the traditional estimation of the PPS so that we could achieve more exact technical efficiency measures which are based upon more realistic frontiers.

When we try to determine the trade-off information that is supposed to be incorporated into the DEA model, we should be careful to ensure that the trade-offs really reflect simultaneous changes in the levels of the corresponding variables without having any effect on the levels of the remaining inputs and outputs that are valid at all units in the technology, and this is because the information is generally used to estimate the whole PPS. As put forward by Podinovski (2004), the relation described by the used trade-offs should not be comparatively challenging, rather it should be used in different units [14]. Therefore, the concept of trade-offs referred to in this context differs from the notion of marginal rates of substitution used in production economics; the latter reflects the exact proportions in which the inputs and outputs of a particular unit on the efficient frontier can change and generally varies from unit to unit.

Since long ago, this trade-off information has been inserted into the DEA models in the form of weight restrictions, both of type AR-I and AR-II (see Podinovski (2004), [14]) for an implementation by modifying the primal envelopment formulation). It should be kept in mind that the reckoning of AR constraints based on either price or trade-off information has a different influence on the obtained efficiency score. While in the latter case, the efficiency score maintains its meaning of technical efficiency measure on the condition that the only objective information about the process is considered. In the former case, no direct information on the production process is considered, rather information on the economic values of the variables is considered so that the resulting efficiency score evaluates both

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allocative and technical efficiency.

While analyzing the Ontario-based branches of a large Canadian bank, Schaffnit et al. (1997) incorporated trade-off information based on standard transaction and maintenance times [15]. Olesen and Petersen (2002) used the probabilistic assurance regions (Olesen and Petersen (1999), [16]) in a different approach in order to incorporate trade-off information into DEA models [17].

For the purpose of improving the DEA models, another approach was suggested based upon the incorporation of production trade-offs in the envelopment DEA models, [14], [18], [3]. These trade-offs represent concurrent changes to the inputs and outputs that are possible technologically. The incorporation of trade-offs in the envelopment models has the same mathematical effect as that caused by weight restrictions in the multiplier forms, that is to say, the resulting DEA models have a better discriminate power.

In this paper, the researchers develop a one-model approach to estimate input congestion of the evaluating DMU with productions trade-offs or, alternatively, under weights restrictions in data envelopment Analysis (DEA). This subject matter and approach have not been so extensively addressed in previous researches carried until now.

This paper contains the following sections: in section 2, a review of the two-model approach to evaluate congestion is presented, section 3 introduces a procedure for practical application of models that incorporate production trade-offs between inputs and outputs or, equivalently, weight restrictions imposed on their dual models as well as the suggested models for assessing the congestion of such models; furthermore, numerical examples are given in section 4. Finally, the conclusions are presented in section 5.

2 The one-model approach

Suppose there are n DMUs, each of them using m inputs to produces outputs. Here we use x_{ij} to represent the level of the $i-$the input $(i = 1, ..., m)$ and y_{rj} the level of the $r-$the output $(r = 1, ..., s)$ from the $j-$the unit, $(j \in J = \{1, ..., n\})$. Cooper et al. (2001a) suggested three-model approaches in order to obtain congestion with BCC (see Banker et al. [13]) model, as follows (two models of model (1) and one model of model (3)) [12]: Therefore we have:

$$
\begin{aligned}
&\phi^* = Max\ \phi + \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)\\
&s.t. \quad \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io}, \qquad i = 1,\ldots,m,\\
&\sum_{j=1}^{n}\lambda_j y_{rj} - s_r^+ = \phi y_{ro}, \quad r = 1,\ldots,s,\\
&\sum_{j=1}^{n}\lambda_j = 1,\\
&\lambda_j \geq 0, \qquad j = 1,\ldots,n,\\
&s_i^- \geq 0, \qquad i = 1,\ldots,m,\\
&s_r^+ \geq 0, \qquad r = 1,\ldots,s,\\
&\phi, \qquad free.
\end{aligned}
\tag{1}
$$

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The symbol $j=0$ represents one of the DMU_o as the DMU_o to be evaluated relative to all data (including the data on DMU_o) where $\varepsilon>0$ is a "non-Archimedean element", which is defined as being smaller than any positive real number. In other words, ε is not a real number. In the standard approach, one tries to avoid assigning a value to ε and for this purpose one can use the following two-stage procedure. Stage one: maximizing ϕ while ignoring the slacks s_i^-, s_r^+, in the objective. Stage two: replacing ϕ with max ϕ^* in (1) and maximizing the sum of the slacks. For an optimal solution $(\lambda^*, \phi^*, s_i^{-*}, s_r^{+*})$ of (1), (here, * is used to represent an optimal value), we set;

$$\hat{x}_{io} = x_{io} - s_i^{-*}, i = 1,...,m \qquad \text{and} \qquad \hat{y}_{ro} = \phi^* y_{ro} + s_r^{+*},\ r = 1,\ldots,s \tag{2}$$

The formula (2), which are taken from (see Banker et al. [13]) are called the "BCC projection formulas" because they project the observed y_{ro} and x_{io} into $\hat{y}_{ro}$ and $\hat{x}_{io}$ on the efficient frontier. As proved in (see Banker et al. [13]). Now, we can use the following model:

$$\begin{aligned}
Max \quad & \sum_{i=1}^{m} \delta_i^- \\
& \sum_{j=1}^{n} x_{ij}\lambda_j - \delta_i^- = x_{io} - s_i^{-*} = \hat{x}_{io}, \qquad i = 1,\ldots,m, \\
s.t. \quad & \sum_{j=1}^{n} y_{rj}\lambda_j = \varphi^* y_{ro} + s_r^{+*} = \hat{y}_{ro}, \qquad r = 1,\ldots,s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& s_i^{-*} \geq \delta_i^-, \\
& \delta_i^- \geq 0, \qquad i = 1,\ldots,m, \\
& \lambda_j \geq 0, \qquad j = 1,\ldots,n.
\end{aligned} \tag{3}$$

In order to obtain the amount of congestion, we can proceed as follows:

$$s_i^{-c^*} = s_i^{-*} - \delta_i^{-*}, i = 1,\ldots,m$$

Where δ_i^{-*} is the model optimal solution of model (3), and (4) is the congestion amount in input $i = 1, ..., m$. In order to evaluate congestion, Cooper *et al.,* (2002) introduced the one-model approach as follows [1]:

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$$
\begin{aligned}
Min \quad & \sum_{i=1}^{m} s_i^{-c} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^{-c} = x_{io}, \quad i = 1,...,m, \\
& \sum_{J=1}^{n} \lambda_j y_{rj} - s_r^{+*} = \phi^* y_{ro}, \quad r = 1,...,s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& s_i^{-c} \geq 0, \quad i = 1,\ldots,m, \\
& \lambda_j \geq 0, \quad j = 1,\ldots,n.
\end{aligned}
\tag{4}
$$

And (λ^*, s_i^{-c*}) is an optimal solution of (4). Now s_i^{-c*} represents the congesting amount of input $i = 1,...,m$. It should be noted that, in fact, model (4) is a combination of three models, i.e. two models of model (1) and one model of model (3). Consider $(\lambda^*, \phi^*, s_i^{-c*}, s_r^{+*})$ the model optimal solution of model (4) in evaluating DMU_o, then s_i^{-c*} is the "congestion amount". Here, however, we can consider (4) as being derived from the following modification of (1):

$$
\begin{aligned}
\phi^* = Max \ & \phi + \varepsilon \left(\sum_{r=1}^{s} s_r^{+} - \sum_{i=1}^{m} s_i^{-c} \right) \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^{-c} = x_{io}, \quad i = 1,...,m, \\
& \sum_{J=1}^{n} \lambda_j y_{rj} - s_r^{+} = \phi y_{ro}, \quad r = 1,...,s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& s_i^{-c} \geq 0, \quad i = 1,\ldots,m, \\
& s_r^{+} \geq 0, \quad r = 1,\ldots,s, \\
& \lambda_j \geq 0, \quad j = 1,\ldots,n, \\
& \phi, \quad free.
\end{aligned}
\tag{5}
$$

2.1 Computation of congestion with productions trade-offs

Here we follow the concepts as presented in Podinovski (2007a, 2007b) and let (P,Q) trade-offs between the inputs and/or outputs designate the possible simultaneous changes occurring in the inputs and outputs in the whole technology [2,3]. Podinovski (2004) has provided many examples of production trade-offs [14].

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Suppose that we can specify k trade-offs:

$$(P_t;Q_t); \qquad t=1,2,...,k. \tag{6}$$

The application of trade-offs (6) in the standard VRS technology results in the expanded technology T_{VRS-TO}, where the abbreviation TO stands for trade-offs, as shown below:

$$T_{VRS-TO}=\left\{\begin{array}{l}(X,Y)\,\big|\,X\geq 0, Y\geq 0, X\geq \sum_{j=1}^{n}\lambda_j X_j+\sum_{t=1}^{k}\pi_t P_t, Y\leq \sum_{j=1}^{n}\lambda_j Y_j+\sum_{t=1}^{k}\pi_t Q_t, \sum_{j=1}^{n}\lambda_j=1\\ ,\lambda_j\geq 0, j=1,...,n; t=1,...,k\end{array}\right\} \tag{7}$$

According to Podinovski (2004) the output radial efficiency of DMU_o is equal to the optimal value ϕ^* of the objective function in the following linear program [14]:

$$\begin{aligned}
&Max \quad \phi\\
&s.t. \quad \sum_{j=1}^{n}\lambda_j X_{ij}+\sum_{t=1}^{k}\pi_t P_t\leq X_{io}, \quad i=1,...,m,\\
&\qquad \sum_{j=1}^{n}\lambda_j Y_{rj}+\sum_{t=1}^{k}\pi_t Q_t\geq \phi Y_{ro}, \quad r=1,...,s,\\
&\qquad \sum_{j=1}^{n}\lambda_j=1,\\
&\qquad \lambda_j\geq 0, \qquad j=1,\ldots,n,\\
&\qquad \pi_t\geq 0, \qquad t=1,\ldots,k,\\
&\qquad \phi, \qquad free.
\end{aligned} \tag{8}$$

The target DMU $(X_o,\phi^* Y_o)$ is a valid member of the PPS T_{VRS-TO} and located on its boundary. Thus, the radial efficiency ϕ^* of DMU_o acts as a radial improvement factor that is technologically feasible for the inputs of DMU_o.

We can say that the introduction of trade-offs (6) into the envelopment models has an equivalent effect as the incorporation of weight restrictions in the dual multiplier forms.

$$u^T Q_t - v^T P_t\leq 0, \quad t=1,\ldots,k \tag{9}$$

Now we can solve the following linear program:

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$$
\begin{aligned}
Max \quad & \sum_{i=1}^{m} d_i + \sum_{r=1}^{s} e_r \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{k} \pi_t P_t + w_i + d_i = x_{io}, \qquad i = 1,...,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{k} \pi_t Q_t - e_r = \phi^* y_{ro}, \qquad r = 1,...,s, \\
& \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{k} \pi_t P_t + w_i \geq 0, \qquad i = 1,...,m, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda, \pi, e, d, w \geq 0.
\end{aligned}
\tag{10}
$$

Model (10) maximizes the sum of residual slacks only on the exact condition that the efficient target thus obtained has only nonnegative inputs. In this stage an entirely efficient target of DMU_o is produced.

Let $(\lambda^*, \pi^*, e^*, w^*, d^*)$ be any optimal solution to model (10). Define

$$
\begin{aligned}
& \sum_{j=1}^{n} \lambda_j^* x_{ij} + \sum_{t=1}^{k} \pi_t^* P_t + w_i^* = \hat{X}_{io} \\
& \sum_{j=1}^{n} \lambda_j^* y_{rj} + \sum_{t=1}^{k} \pi_t^* Q_t = \hat{y}_{ro}
\end{aligned}
\tag{11}
$$

Obviously, DMU $(\hat{x}_{io}, \hat{y}_{ro})$ is a member of technology T_{VRS-TO}.

Theorem 1. The DMU $(\hat{x}_{io}, \hat{y}_{ro})$ is pareto-efficient in technology T_{VRS-TO}.

According to Theorem 1, if $\phi^* = 1$ and optimal vectors e^* and d^* are zero vectors, DMU_o coincides with DMU $(\hat{x}_{io}, \hat{y}_{ro})$ and is therefore efficient. Otherwise, DMU_o is inefficient and $(\hat{x}_{io}, \hat{y}_{ro})$ may be considered as its efficient target.

According to section 2, we can first apply (2) in order to obtain the BCC-projection of a DMU_o, that is, $(\hat{x}_{io}, \hat{y}_{ro})$, which were presented to compute the congestion. By the same token, under weight restrictions, our goal is to find the radial target of a DMU_o using model (8), which then will be used to obtain the correspondent efficient target in model (10) according to (11) and (12). Now through incorporating the mentioned efficient target in Model (13) we are going to calculate the congestion.

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$$
\begin{aligned}
Min \quad & \sum_{i=1}^{m} s_i^{-c} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + \sum_{t=1}^{k} \pi_t P_t + s_i^{-c} = x_{io}, \qquad i = 1,\ldots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \sum_{t=1}^{k} \pi_t Q_t = \phi^* y_{ro} + e_r^*, \quad r = 1,\ldots,s, \qquad (13) \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \qquad j = 1,\ldots,n, \\
& \pi_t \geq 0, \qquad t = 1,\ldots,k, \\
& s_i^{-c} \geq 0, \qquad i = 1,\ldots,m.
\end{aligned}
$$

The congesting amount of input $i = 1,\ldots,m$, is then represented by $s_i^{-c^*}$ in accordance with the following theorems:

Theorem 2. Congestion is present if and only if in an optimal solution $(\lambda^*, \pi_t^*, \phi^*, s_i^{-c^*}, e_r^*)$ of (9), at least one of the following two conditions is satisfied:

$\phi^* > 1$ and there is at least one $s_i^{-c^*} > 0, (i = 1,\ldots,m)$.

There exists at least one $e_r^* > 0, (r = 1,\ldots,s)$ and at least one $s_i^{-c^*} > 0, (i = 1,\ldots,m)$.

Obviously, in both cases DMU_o is inefficient. In other words, if DMU_o has input congestion, then it is inefficient while the reverse is not true.

Theorem 3. Suppose $(\lambda^*, \pi_t^*, \phi^*, s_i^{-c^*}, e_r^*)$ is an optimal solution of (13). Then,

(i). If $\phi^* > 1$, then DMU_o is inefficient.

(ii). If there exists at least one $e_r^* > 0, (r = 1,\ldots,s)$, then DMU_o is inefficient.

(iii). If there exists at least one $s_i^{-c^*} > 0, (i = 1,\ldots,m)$, then DMU_o is inefficient because congestion is present.

(iv). If $\phi^* = 1$; $s_i^{-c^*} = 0, (i = 1,\ldots,m)$, and $e_r^* = 0, (r = 1,\ldots,s)$, then DMU_o is on a frontier.

One could say that this model could be considered as part of a two-stage procedure which is to a great extent similar to the model explained earlier when discussing the non-Archimedean element $\varepsilon > 0$ in (1). But here we can assume (13) has been derived from the following:

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$$\phi^* = Max\ \phi + \varepsilon\left(\sum_{r=1}^{s} e_r - \sum_{i=1}^{m} s_i^{-c}\right)$$

$$\begin{aligned} s.t. \quad & \sum_{j=1}^{n}\lambda_j x_{ij} + \sum_{t=1}^{k}\pi_t P_t + s_i^{-c} = x_{io}, \qquad i = 1,\ldots,m, \\ & \sum_{j=1}^{n}\lambda_j y_{rj} + \sum_{t=1}^{k}\pi_t Q_t = \phi y_{ro} + e_r, \quad r = 1,\ldots,s, \\ & \sum_{j=1}^{n}\lambda_j = 1, \\ & \lambda_j \geq 0, \qquad j = 1,\ldots,n, \\ & \pi_t \geq 0, \qquad t = 1,\ldots,k, \\ & s_i^{-c} \geq 0, \qquad i = 1,\ldots,m, \\ & e_r \geq 0, \qquad r = 1,\ldots,s. \end{aligned} \tag{14}$$

2.2 Empirical example

Table 1 shows six hypothetical university departments that we used in order to illustrate the case. These departments are based on two inputs, 'Teaching staff' and 'Research staff', as well as three outputs, 'Undergraduate students', 'Master students' and 'Publications'.

Table 1 The data set

	Input		Output		
Department	**Teaching staff**	**Research staff**	**Undergraduate students**	**Master students**	**Publications**
A	100	70	1540	154	154
B	120	123	1408	186	186
C	50	20	690	59	59
D	67	17	674	73	73
E	98	20	1686	197	197
F	76	12	982	63	63

Source of data: Podinovski (2007b),[3]

Table 2 Results of the standard BCC model (1)

Department	s_1^{-*}	s_2^{-*}	s_1^{+*}	s_2^{+*}	s_3^{+*}	ϕ^*
A	0	0	0	0	0	1
B	$\frac{209484}{9731}$	$\frac{887343}{9731}$	$\frac{2307612}{9731}$	0	0	$\frac{9775}{9731}$
C	0	0	0	0	0	1
D	0	0	0	0	0	1
E	0	0	0	0	0	1
F	0	0	0	0	0	1

The first and last columns of Table (2) show the departments and the value of ϕ^*, respectively, and other columns indicate optimal slack solutions according to Model (1).

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Based on the results of Table (2), only department B is inefficient. Therefore, we have the following BCC-projection:

$$(\hat{x}_1 = \frac{958236}{9731}, \hat{x}_2 = \frac{309570}{9731}, \hat{y}_1 = \frac{16070812}{9731}, \hat{y}_2 = \frac{1818150}{9731}, \hat{y}_3 = \frac{224825}{9731})$$

If we incorporate the BCC-projection into Model (3), the congesting inputs and their amounts are obtained as follows:

$$s_1^{-c*} = \frac{209484}{9731}, \quad s_2^{-c*} = \frac{887343}{9731}$$

In this part, we want to show how production trade-offs can be assessed in real technologies and, particularly, in this case.
In the present example, different production trade-offs are applicable, some of which are briefly presented here. (For more information, one can refer to [3])

1. Firstly, we suppose that in teaching master students, we need at most twice the amount of resources used to teach undergraduate students. In other words, if in any department the number of undergraduate students is reduced by 2, and the number of master students is increased by 1, then that department cannot claim extra resources-i.e. teaching and research staff. Furthermore, this change could barely have any influence on the outputs of the research.
$$P_1 = (0,0)^T, \qquad Q_1 = (-2,1,0)^T \tag{15}$$
One can read it as the following: the situation where two undergraduate students are out and one master student is in is practical on the condition that other variables do not change.
2. In the situation where the number of undergraduate students is increased by 1 and the number of master students is reduced by 1, we would not need any extra resources. The related trade-off can be shown as follows:
$$P_2 = (0,0)^T, \qquad Q_2 = (1,-1,0)^T \tag{16}$$
3. If one less paper is published in a year, the time that would be otherwise allocated to that should be enough to teach two extra undergraduate students:
$$P_3 = (0,0)^T, \qquad Q_3 = (2,0,-1)^T \tag{17}$$
4. For each teaching position which is introduced, the number of undergraduate students can increase at least by five; that mounts to 25 on the whole. Therefore, we can formulate the following trade-off, which is apparently justifiable:
$$P_4 = (5,-1)^T, \qquad Q_4 = (25,0,0)^T \tag{18}$$
5. Finally, suppose the situation where one teaching post is replaced by one research position. Due to the fact that generally the research staffs do not teach, we should take into consideration a harmful effect on students while examining this trade-off. No more than 20 undergraduate students are supposed to be subtracted from the students of any department, even in the worst conditions. Also, since the research staff is supposed to

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publish more papers than the teaching staff, one may expect publications to increase by at least 0.3 papers a year. This situation may result in the following trade-off, which has simultaneously two inputs and two outputs.

$$P_5 = (-1,-1)^T, \qquad Q_5 = (-20,0,0.3)^T \tag{19}$$

Table 3 the Computational results of model (13) for determining input congestion with productions trade-offs (15), (16), (17), (18), and (19)

Department	d_1^*	d_2^*	e_1^*	e_2^*	e_3^*	ϕ^*	s_1^{-c*}	s_2^{-c*}
A	0	0	0	0	0	1	0	0
B	0	$\frac{2255769}{24785}$	0	0	0	$\frac{17639}{14871}$	0	$\frac{2255769}{24785}$
C	0	0	0	0	0	1	0	0
D	0	0	0	0	0	1	0	0
E	0	0	0	0	0	1	0	0
F	0	0	0	0	0	1	0	0

Computational results for the models (8) and (10). The computational results of the model (13), one-model approach, for determining input congestion with productions trade-offs (15), (16), (17), (18), (19) are shown in Table (3), according to the above table, only department B is non-radial efficient. d_i and e_r^* $(i = 1,...,m; r = 1,...,s)$ are obtained by solving Model (10). We can calculate $(\hat{X}_o, \hat{Y}_o)$ and incorporate it in Model (13), the congesting inputs and their amounts. As it was seen, when production trade-offs are applied, the values of the congestions in the inputs vary.

3 Conclusions

In this paper, we computed congestion in DEA models with productions trade-offs and weight restrictions. So far, many approaches have been presented to evaluate congestion, but the evaluation of congestion under weights restrictions, or equivalently with production trade-offs, is still a new topic. The researchers compared computational results of congestion in original DEA models with the computational results of the DEA models under weight restrictions by using a numerical example. Since weight restrictions are imposed in DEA models, it is not surprising that the numerical results are not similar to those of the original DEA models. Finally, it is recommended that researchers apply this method to other types of weights restrictions in DEA. and as shown in table (3), the amount of congestion of the first input in department B has been reduced to zero with productions trade-offs and weight restrictions; it means that by applying appropriate trade-offs, a manager can reduce the amount of congestion to zero.

References

1. Cooper, W. W., Deng, H., Huang, Z.M., Li S. X., (2002). A one-model approach to congestion in data envelopment analysis. Socia-Economic Planning Sciences, 36, 231-238.
2. Podinovski, V. V., (2007)a. Computation of Efficient Targets in DEA Models with Production Trade-Offs and Weight Restrictions. European Journal of Operational Research, 181, 586-591.

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3. Podinovski, V. V., (2007) b. Improving data envelopment analysis by the use of production trade-offs. Journal of the Operational Research Society, 58, 1261-1270.
4. Charnes, A., Cooper, W. W., Rhodes, E., (1978). Measuring the efficiency of decision making units. European Journal of Operational Research, 2, 429-444.
5. Färe, R., Ggrosskopf, S., (1983). Measuring congestion production. Zeitschrift fur Nationalokonomic, 257-271.
6. Shephard, R. W., (1970).Theory of Cost and Production Functions. Princeton University Press, Princeton.
7. Färe, R., Ggrosskopf, S., Lovell, C. A. K., (1985). The measurement of efficiency of production. Boston: Kluwer-Nijhoff Publishing Co.
8. Cooper, W. W., Thompson, R. G., Thrall, R. M., (1996). Introduction: Extensions and new developments in DEA. Annals of Operations Research, 66, 3–45.
9. Cooper, W. W., Gu, B. S., Li, S. L., (2001c). Comparisons and evaluation of alternative approaches to the treatment of congestion in DEA. European Journal of Operational Research, 132, 62–74.
10. Brockett, P. L., Cooper, W. W., Shin, H. C., Wang, Y., (1978). Inefficiency and congestion in Chinese production before and after the economic reforms 1998. Socio-Economic Planning Sciences, 32, 1–20.
11. Cooper, W. W., Seiford, L. M., Zhu, J., (2000b). A unified additive model approach for evaluating inefficiency and congestion. Socio- Economic Planning Sciences, 34, 1–26.
12. Cooper, W. W., Deng, H., Gu, B. S., Li, S. L., Thrall, R. M., (2001)a. Using DEA to improve the management of congestion in Chinese industries 1981–1997. Socio-Economic Planning Sciences, 35, 227–242.
13. Banker, R. D., Charnes, A., Cooper, W. W., (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. Manag Sci, 30, 1078 –92.
14. Podinovski, V. V., (2004). Production trade-offs and weight restrictions in data envelopment analysis. Journal of the Operational Research Society, 55, 1311-1322.
15. Schaffnit, C., Rosen, D., Paradi, J. C., (1997).Best practice analysis of bank branches: an application of DEA in a large Canadian bank. European Journal of Operational Research, 98, 269–289.
16. Olesen, O. B., Petersen N. C., (1999). Probabilistic bounds on the virtual multipliers in data envelopment analysis: polyhedral cone constraints. J Product Anal, 12(2), 103–33.
17. Olesen, O. B., Petersen, N. C., (2002). The use of data envelopment analysis with probabilistic assurance regions for measuring hospital efficiency. J Product Anal, 17(1/2), 83–109.
18. Podinovski, V. V., (2005). The explicit role of weight bounds in models of data envelopment analysis. Journal of the Operational Research Society, 56, 1408-1418.
19. Jahanshahloo, G. R., Khodabakhshi, M., (2004). Suitable combination of inputs for improving outputs in DEAwith determining input congestion. Considering textile industry of China, 151, 263-273.
20. Khodabakhshi, M., (2009). A one-model approach based on relaxed combinations of inputs for evaluating input congestion in DEA. Journal of Computational and Applied Mathematics, 230, 443-450.
21. Cooper, W. W., Seiford, L. M., Tone, K., (2000). Data Envelopment Analysis. Kluwer Academic Publishers, Dordrecht.

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