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Dynamic DEA: A Hybrid Measure Approach

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Abstract In the real world applications, there are some situations where inputs and outputs are time-dependent and are affected during the production periods. Capital stock can be seen as an effective instance in such occasions. In order to handling long-time planning, dynamic structure was proposed in efficiency evaluation. In this framework, there are some of the inputs and outputs change proportionally, which is called radially and the others change non-radially. Hence, for efficiency evaluation a combination of the radial and non-radial approaches, the hybrid model is applied. This paper goals to extend the hybrid measure in dynamic structure of Data Envelopment Analysis(DEA), also efforts to evaluate the overall efficiency of Decision Making Units(DMUs) with both radial and non-radial inputs and outputs for the whole terms as well as each term efficiencies.

Keywords: DMU (Decision Making Unit), Efficiency, DEA (Data Envelopment Analysis), HD-DEA (Hybrid Dynamic DEA), Carry-Over.

1 Introduction

DEA is shown as a non-parametric technique for measuring the relative efficiency of DMUs and their ranking. This technique has been pioneered by Charnes, Cooper and Rhodes [1] and later extended by Banker et.al [2]. Following them, several models were introduced in the DEA literature such as the Slack-Based Measure (SBM) model proposed by Tone [3], the additive model proposed by Charnes et al. [4], the Russell measure model proposed by Russell [5] and the Enhanced Russell Measure (ERM) proposed by Pastor et al. [6]. Also for evaluating the efficiency of DMUs, when some of the inputs or outputs change proportionally (the change of these inputs or outputs are radially) and some others of the inputs or outputs change non-radially, the hybrid model is proposed by Tone [7], which is combination of the non-radial and radial approaches. The presented DEA models have been widely used to measure efficiency of DMUs in the static situation. Despite their widespread popularity, the classical DEA model and its extensions operate under the implicit assumption that production technologies are static and independent across time. Note that, important factors such as inter-temporal effect and carry-over activities are ignored in efficiency measuring processes and

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only focus on the separate time period, independently. Also in the actual business world, we encounter with long time planning and investments and situations where input and output levels are time-dependent. For example, capital stock which is effective on the output levels during the production periods. Therefore, classic or static DEA models cannot be used. To overcome this problem in DEA, in the first time the window analysis proposed by Klopp [8]. Based on Malmquist [9], Fare et al. [10] developed the Malmquist index in the DEA framework. These models do not account for the effect of carry-over activities between two consecutive terms. Therefore, the Dynamic DEA (D-DEA) model was proposed by Fare and Grosskopf [11]. In the proposed model, the impact of interconnecting activities intended in the evaluating of efficiency of DMUs. Dynamic models allow a decision in one period to influence the outcomes in other periods and this time interdependence is the essence of a dynamic model. After Fare and Grosskopf [11], a lot of works was done on the dynamic DEA such as Tone and Tsutsui [12], Amirteimoori [13], Chen [14], Nemoto and Goto [15, 16], Park and Park [17], Tone and Tsutsui [12], Sueyoshi and Sekitani [18] and Chang et al. [19]. Up to now, some of the DEA models are developed in D-DEA structure. For example, the SBM model is extended in D-DEA structure by Tone and Tsutsui [21], also the D-DEA additive model proposed by Sanei et al. [20]. In D-DEA framework, there are situations that some of the inputs or outputs change radially and others change non-radially, so the efficiency evaluation in these situations is necessary. Therefore in this paper, the hybrid model extension in D-DEA structure is introduced and is called the Hybrid Dynamic DEA (HD-DEA) model. We expand the HD-DEA model that can evaluate the overall efficiency of DMUs with radial and non-radial inputs or outputs for the whole terms as well as the term efficiencies. According to [7], the carry-overs or links are categorized into four types, i.e. desirable (good), undesirable (bad), discretionary (free) and non-discretionary (fixed) which reflect actual characteristics of carry-over activities and will explain in section 3. With these four categories, demands of researchers and practitioners are supplied correctly and properly. This paper is organized as follows. Hybrid measure in DEA is reviewed in Section 2. Dynamic structure and links and their characteristics are presented in section 3. The Hybrid measure in dynamic DEA framework and its projection are presented in section 4. Conclusions appear in section 5.

2 Hybrid Measure in Data Envelopment Analysis (DEA)

As far as we are aware, the DEA models are categorized to radial and non-radial models. The radial models were displayed by the CCR model [1] with constant return to scale and later the BCC model [2] with variable return to scale. About its deficiency can be said that the non-radial input or output slacks are not considered. The non -radial models such as the SBM model [3], the additive model [4], the Russell measure model [5] and the Enhanced Russell Measure (ERM) [6] have a common weakness which the radial characteristics of inputs or outputs are not considered. Therefore, with integration these models in a unified framework, the hybrid measure of efficiency is represented by Tone [7]. As it is known, there are differences in the characterization of input or output items. In the radial input or output, the change is proportionally, while in the non-radial input or output, the change is out of proportion. For more details, suppose that $(x_1, x_2, ..., x_m)$ are radial inputs and $(x_{m+1}, x_{m+2}, ..., x_n)$ are non-radial inputs. According to the proportionate change in the radial inputs, we have $(\alpha x_1, \alpha x_2, ..., \alpha x_m)$ with $\alpha \geq 0$. About the non-radial inputs, since the change is

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not proportionate, therefore each input change non-radially based on its slack. According to [7], for representation the hybrid measure, assumed that the input and output data matrix be $X \in R_+^{m*n}$ and $Y \in R_+^{s*r}$. Which n, m and s designate the number of DMUs, inputs and outputs respectively. The input matrix X can be decomposed into the radial and non-radial part $X^R \in R_+^{m_1 \times n}$ and $X^{NR} \in R_+^{m_2 \times n}$ respectively, with $m = m_1 + m_2$ and also, the output matrix Y is decomposed into the radial and non-radial part $Y^R \in R_+^{s_1 \times n}$ and $Y^{NR} \in R_+^{s_2 \times n}$ respectively, with $s = s_1 + s_2$ as follow:

$$X = (X^R, X^{NR})^t \qquad Y = (Y^R, Y^{NR})^t$$

Note that, the input and output data are positive, i.e. $X > 0$ and $Y > 0$. The production possibility set (PPS) P is defined as:

$$P = \left\{ (x, y) \middle| x \geq X\lambda \ , y \leq Y\lambda \ , \lambda \geq 0, \lambda \in R^n \right\}$$

Suppose that DMU_o is under evaluated unit with $(x_o, y_o) = (x_o^R, x_o^{NR}, y_o^R, y_o^{NR}) \in P$. According to the different characteristic of inputs and outputs we have:

$$\begin{aligned} \theta x_o^R &= X^R \lambda + s^{R-} \\ x_o^{NR} &= X^{NR} \lambda + s^{NR-} \\ \phi y_o^R &= Y^R \lambda - s^{R+} \\ y_o^{NR} &= Y^{NR} \lambda - s^{NR+} \end{aligned} \tag{1}$$

With $\theta \leq 1, \varphi \geq 1, \lambda, s^{R-}, s^{NR-}, s^{R+}, s^{NR+} \geq 0.$ Input slacks are represented by the vectors $s^{R-} \in R^{m_1}$ and $s^{NR-} \in R^{m_2}$ which these are corresponding to surplus for the radial and non-radial inputs, respectively. Similarly, output slacks are displayed by the vectors $s^{R+} \in R^{s_1}$ and $s^{NR+} \in R^{s_2}$ which these are corresponding to shortages for the radial and non-radial outputs, respectively. Based on [7], the hybrid model is defined as follows:

$$\rho_o = \min \ \frac{1 - \left(\frac{m_1}{m}\right)(1-\theta) - \frac{1}{m}\sum_{i=1}^{m_2} \frac{s_i^{NR-}}{x_{io}^{NR}}}{1 + \left(\frac{s_1}{s}\right)(\phi - 1) - \frac{1}{s}\sum_{r=1}^{s_2} \frac{s_i^{NR+}}{y_{ro}^{NR}}}$$

$s.t.$

$$\begin{aligned} \theta x_o^R &\geq \lambda X^R \\ x_o^{NR} &= \lambda X^{NR} + s^{NR-} \\ \phi y_o^R &\leq \lambda Y^R \\ y_o^{NR} &= \lambda Y^{NR} - s^{NR+} \\ \theta \leq 1, \phi &\geq 1, \ \lambda, \ s^{NR-}, s^{NR+} \geq 0 \end{aligned} \tag{2}$$

The objective function value of (2) is called ρ_O^* which is unit invariant, i.e. invariant with respect to the measurement units of the data. Note that, the objective function is not affected by s^{R--} and s^{R+} directly, which is reflecting free disposability of these radial slacks. Let an optimal solution for the model (2) is $(\theta^*, \varphi^*, \lambda^*, s^{NR-*}, s^{NR+*})$. Therefore according to [7], DMU_o is hybrid-efficient if and only if $\rho_O^* = 1$, In other words, $\theta^* = 1, \phi^* = 1$ and $s^{NR-*} = 0, s^{NR+*} = 0$.

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3 Dynamic Structure

In Data Envelopment Analysis (DEA), for evaluating efficiency changes over time when inter-temporal effects and carry-over activities are existed in long time planning between two consecutive terms, the dynamic DEA structure is proposed. In dynamic situations, a panel data through terms 1 to T is considered as exhibited in Fig. 1.

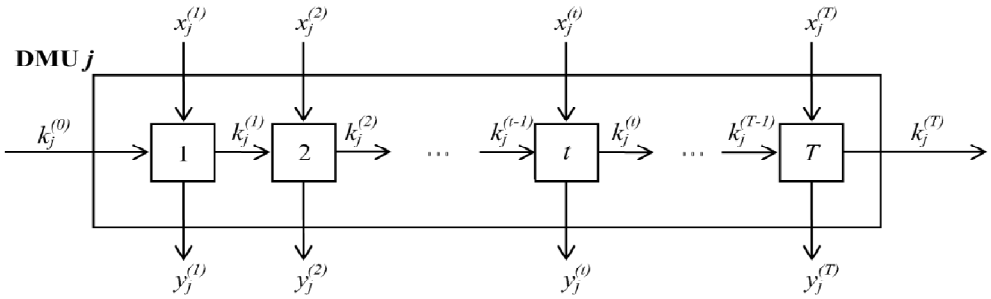


Fig. 1 Dynamic DEA Structure

In every term $t = 1,...,T$ each DMU has its respective inputs and outputs along with the carry-over (link) to the next term $t + 1$. The explicit difference between the dynamic DEA and the ordinary DEA is the existence of carry-overs that connect two consecutive terms. According to [20], carry-over activities are classified into four categories as follows:

- Desirable (good) carry-over. For example, profit or retained earnings and net earned surplus carried to the next term are of the desirable carry-over category. Note that, desirable links are treated as outputs and link value is restricted to be not less than the observed one and also, comparative shortage of links in this category is accounted as inefficiency.
- Undesirable (bad) carry-over. For example, loss carried forward, bad debt and dead stock are of the undesirable carry-over category. Undesirable links are treated as inputs and its value is restricted to be not greater than the observed one and also, comparative excess in links in this category is accounted as inefficiency.
- Discretionary (free) carry-over. This carry-over can be handled freely by each DMU and its value can be increased or decreased from the observed one. The deviation from the current value is not directly reflected in the efficiency evaluation and its value has an indirect effect on the efficiency score.

Non-discretionary (fixed) carry-over. This carry-over is beyond the control of DMU and its value is fixed at the observed level. Also, similarly to free carry-over, fixed carry-over affects the efficiency score indirectly.

Based on the mentioned structures, in the following section, the hybrid measure on dynamic DEA are discused on detail.

4 Hybrid Measure in Dynamic DEA

Assume that, there are n DMUs $(j = 1,...,n)$ over $t = 1,...,T$ term, so that at each term, DMUs have common m inputs that m_1 inputs are radial and m_2 inputs are non-radial with $m_1 + m_2 = m$ and ρ non-discretionary (fixed) inputs. Also at each term, DMUs have common s outputs that s_1 outputs are radial and s_2 outputs are n1on-radial with $s_1 + s_2 = s$

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and p non-discretionary (fixed) outputs. Let $x_{ijt}^{R}(i=1,...,m_1), x_{ijt}^{NR}(i=1,...,m_2)$, $x_{ijt}^{fix}(i=1,...,\rho), y_{rjt}^{R}(r=1,...,s_1)$ and $y_{rjt}^{fix}(r=1,...,p)$, denote radial input, non-radial input, nondiscretionary input, radial output, non-radial output and non-discretionary output values of DMU_j at term t, respectively. The four links or carry-overs contain good, bad, free and fix carry-overs are symbolized as , z^g, z^b, z^f and z^{fix} respectively, such that the notations $z_{ijt}^{gR}(i=1,....,n_1 g)$, $z_{ijt}^{gNR}(i=1,....,n_2 g)$, $z_{ijt}^{bR}(i=1,....,n_1 b)$, $z_{ijt}^{gR}(i=1,....,n_1 g)$ denote radial good link, non-radial good link, radial bad link and non-radial bad link values of DMU_j at term t, respectively. For all $t=1,...,T$ the production possibilities $\{x_{it}^{R}\}, \{x_{it}^{NR}\}, \{x_{it}^{fix}\}$, $\{y_{rt}^{R}\}, \{y_{rt}^{NR}\}, \{y_{rt}^{fix}\}, \{z_{it}^{gR}\}, \{z_{it}^{gNR}\}, \{z_{it}^{bR}\}, \{z_{it}^{bNR}\}$, $\{z_{it}^{f}\}$ and $\{z_{it}^{fix}\}$ can be defined as follows:

$$
\begin{aligned}
&\theta_t x_{it}^{R} \geq \sum_{j=1}^{n} \lambda_j^t x_{ijt}^{R} \ (i=1,...,m_1, \forall t) \\
&x_{it}^{NR} \geq \sum_{j=1}^{n} \lambda_j^t x_{ijt}^{NR} \ (i=1,...,m_2, \forall t) \\
&x_{it}^{fix} = \sum_{j=1}^{n} \lambda_j^t x_{ijt}^{fix} \ (i=1,...,\rho, \forall t) \\
&\varphi_t y_{rt}^{R} \geq \sum_{j=1}^{n} \lambda_j^t y_{rjt}^{R} \ (r=1,...,s_1, \forall t) \\
&y_{rt}^{NR} \leq \sum_{j=1}^{n} \lambda_j^t y_{rjt}^{NR} \ (r=1,...,s_2, \forall t) \\
&y_{rt}^{fix} = \sum_{j=1}^{n} \lambda_j^t y_{rjt}^{fix} \ (r=1,...,p, \forall t) \\
&\phi_t z_{it}^{gR} \leq \sum_{j=1}^{n} \lambda_j^t z_{rjt}^{gR} \ (i=1,...,n_1 g, \forall t) \\
&z_{it}^{gNR} \leq \sum_{j=1}^{n} \lambda_j^t z_{rjt}^{gNR} \ (i=1,...,n_2 g, \forall t) \\
&\varphi_t z_{it}^{bR} \geq \sum_{j=1}^{n} \lambda_j^t z_{rjt}^{bR} \ (i=1,...,n_1 b, \forall t) \\
&z_{it}^{bNR} \geq \sum_{j=1}^{n} \lambda_j^t z_{rjt}^{bNR} \ (i=1,...,n_2 b, \forall t) \\
&z_{it}^{f} : free \qquad (i=1,...,nf, \forall t) \\
&z_{it}^{fix} = \sum_{j=1}^{n} \lambda_j^t z_{ijt}^{fix} \ (i=1,...,nfix, \forall t) \\
&\lambda_j^t \geq 0 \qquad (j=1,...,n, \forall t) \\
&\sum_{j=1}^{n} \lambda_j^t = 1 \quad (t=1,...,T)
\end{aligned}
\tag{3}
$$

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The last constraint corresponds to the variable returns to scale assumption. With deletion this constraint, the constant return to scale is satisfied. The carry-overs continuity between term t and $t+1$ can be satisfied in the following conditions:

$$\sum_{j=1}^{n} z_{ijt}^{\alpha} \lambda_j^t = \sum_{j=1}^{n} z_{ijt}^{\alpha} \lambda_j^{t+1} \quad (\forall i,\ t = 1,...,T-1) \tag{4}$$

The symbol α stands for *good*, *bad*, *free* or *fix* carry-over. This constraint connects activities of terms term t and $t+1$, is very important for dynamic DEA models. The overall efficiency of DMU_o has a value between 0 and 1 and is evaluated by solving the following program with the variables $\theta_t, \psi_t, \phi_t, \varphi_t, \lambda^t, s_t^{-NR}, s_t^{bNR}, s_t^{+NR}, s_t^{gNR}, s_t^{f}$. The objective function has the following format.

$$\sigma^* = \min \frac{\frac{1}{T}\sum_{t=1}^{T} \omega^t \left[1 - \frac{m_1}{m}(1-\theta_t) - \frac{n_1 b}{nb}(1-\psi_t) - \frac{1}{m}\sum_{i=1}^{m_2} \frac{s_{it}^{-NR}}{x_{iot}^{NR}} - \frac{1}{nb}\sum_{i=1}^{n_2 b} \frac{s_{it}^{bNR}}{z_{iot}^{bNR}} \right]}{\frac{1}{T}\sum_{t=1}^{T} \omega^t \left[1 + \frac{s_1}{s}(\phi_t - 1) - \frac{n_1 g}{ng}(\varphi_t - 1) + \frac{1}{s}\sum_{r=1}^{s_2} \frac{s_{it}^{+NR}}{y_{rot}^{NR}} + \frac{1}{ng}\sum_{i=1}^{n_2 g} \frac{s_{it}^{gNR}}{z_{iot}^{gNR}} \right]}$$

The constraint under which the defined objective gets minimized are as follows:

$$\begin{aligned}
&\theta_t x_{iot}^{R} \geq \sum_{j=1}^{n} \lambda_j^t x_{ijt}^{R} \ (i = 1,...,m_1, \forall t) \\
&x_{iot}^{NR} = \sum_{j=1}^{n} \lambda_j^t x_{ijt}^{NR} + s_{it}^{-NR} \ (i = 1,...,m_2, \forall t) \\
&x_{iot}^{fix} = \sum_{j=1}^{n} \lambda_j^t x_{ijt}^{fix} \ (i = 1,...,\rho, \forall t) \\
&\varphi_t y_{rot}^{R} \geq \sum_{j=1}^{n} \lambda_j^t y_{rjt}^{R} \ (r = 1,...,s_1, \forall t) \\
&y_{rot}^{NR} = \sum_{j=1}^{n} \lambda_j^t y_{rjt}^{NR} - s_{it}^{+NR} \ (r = 1,...,s_2, \forall t) \\
&y_{rot}^{fix} = \sum_{j=1}^{n} \lambda_j^t y_{rjt}^{fix} \ (r = 1,...,p, \forall t) \\
&\phi_t z_{iot}^{gR} \leq \sum_{j=1}^{n} \lambda_j^t z_{rjt}^{gR} \ (i = 1,...,n_1 g, \forall t) \\
&z_{iot}^{gNR} = \sum_{j=1}^{n} \lambda_j^t z_{rjt}^{gNR} - s_{it}^{gNR} \ (i = 1,...,n_2 g, \forall t) \\
&\varphi_t z_{iot}^{bR} \geq \sum_{j=1}^{n} \lambda_j^t z_{rjt}^{bR} \ (i = 1,...,n_1 b, \forall t) \\
&z_{iot}^{bNR} = \sum_{j=1}^{n} \lambda_j^t z_{rjt}^{bNR} + s_{it}^{bNR} \ (i = 1,...,n_2 b, \forall t) \\
&z_{iot}^{f} = \sum_{j=1}^{n} \lambda_j^t z_{ijt}^{fix} + s_{it}^{f} \qquad (i = 1,...,nf, \forall t)
\end{aligned} \tag{5}$$

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$$z_{iot}^{fix} = \sum_{j=1}^{n} \lambda_j^t z_{ijt}^{fix} \ (i = 1,...,nfix, \forall t)$$

$$\lambda_j^t \geq 0 \qquad (j = 1,...,n, \forall t)$$

$$\sum_{j=1}^{n} \lambda_j^t = 1 \quad (t = 1,...,T)$$

$$\sum_{j=1}^{n} \lambda_j^t z_{ijt}^{\alpha} = \sum_{j=1}^{n} \lambda_j^{t+1} z_{ijt}^{\alpha} \quad (\forall i,\ t = 1,...,T-1)$$

$$\lambda_j^t \geq 0$$

$$\theta_t, \psi_t \leq 1 \quad \varphi_t, \phi_t \geq 1$$

$$s_{it}^{-NR}, s_{it}^{+NR}, s_{it}^{gNR}, s_{it}^{bNR} \geq 0$$

$s_{it}^{-NR}, s_{it}^{+NR}, s_{it}^{gNR}, s_{it}^{bNR}$ and s_{it}^{f} are slack variables denoting input excess, output shortfall, link shortfall, link excess and link deviation, respectively. Note that, ω^t is a weight of term t which is determined according to the term importance and is satisfied:

$$\sum_{t=1}^{T} \omega^t = T .$$

Model (5) is an extension of the hybrid measure [7] and its objective function deals with excesses in both input resources and undesirable (bad) links and shortfalls in both output products and desirable (good) links in a single unified scheme.

Definition 1: The DMU_o is hybrid efficient if and only if $\sigma^* = 1 = 1$, i.e., $\theta_t^* = \psi_t^* = \phi_t^* = \varphi_t^* = 1$ and $s_t^{-NR*} = s_t^{+NR*} = s_t^{gNR*} = s_t^{bNR*} = 0$.

Given the optimal solution of the above model $\theta_o^{t*}, \psi_o^{t*}, \phi_o^{t*}, \varphi_o^{t*}, \lambda_o^{t*}, s_{ot}^{-NR}*, s_{ot}^{bNR*}, s_{ot}^{+NR*}, s_{ot}^{gNR*}, s_{ot}^{f*}$, the efficiency of DMU_o for $t = 1,...,T$ can

$$\sigma_o^* = \frac{\left[1 - \frac{m_1}{m}(1-\theta_{ot}^*) - \frac{n_1 b}{nb}(1-\psi_{ot}^*) - \frac{1}{m}\sum_{i=1}^{m_2} \frac{s_{iot}^{-NR*}}{x_{iot}^{NR}} - \frac{1}{nb}\sum_{i=1}^{n_2 b} \frac{s_{it}^{bN*R}}{z_{iot}^{bNR}}\right]}{\left[1 + \frac{s_1}{s}(\phi_{ot}^* - 1) - \frac{n_1 g}{ng}(\varphi_{ot}^* - 1) + \frac{1}{s}\sum_{r=1}^{s_2} \frac{s_{iot}^{+N*R}}{y_{rot}^{NR}} + \frac{1}{ng}\sum_{i=1}^{n_2 g} \frac{s_{iot}^{gNR*}}{z_{iot}^{gNR}}\right]}$$

According to the Definition 1, when $\sigma_o^* \neq 1$ DMU_o is called hybrid-inefficient. Equipped with optimal solutions $\theta_o^{t*}, \psi_o^{t*}, \phi_o^{t*}, \varphi_o^{t*}, \lambda_o^{t*}, s_{ot}^{-NR}*, s_{ot}^{bNR*}, s_{ot}^{+NR*}, s_{ot}^{gNR*}, s_{ot}^{f*}$, the hybrid projection of hybrid-inefficient DMU_o is given by:

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$$
\begin{aligned}
&\theta_{ot}^{*} x_{iot}^{R} = \bar{x}_{iot}^{R} \qquad (i = 1,...,m_1, \forall t) \\
&x_{iot}^{NR} - s_{iot}^{-NR*} = \bar{x}_{iot}^{NR} \qquad (i = 1,...,m_2, \forall t) \\
&x_{iot}^{fix} = \bar{x}_{iot}^{fix} \qquad (i = 1,...,\rho, \forall t) \\
&\varphi_{ot}^{*} y_{iot}^{R} = \bar{y}_{rot}^{R} \qquad (r = 1,...,s_1, \forall t) \\
&y_{rot}^{NR} + s_{rot}^{+NR*} = \bar{y}_{rot}^{NR} \qquad (r = 1,...,s_2, \forall t) \\
&y_{rot}^{fix} = \bar{y}_{rot}^{fix} \qquad (r = 1,...,p, \forall t) \\
&z_{iot}^{gNR} + s_{iot}^{gNR*} = \bar{z}_{iot}^{gNR} \qquad (i = 1,...,n_2 g, \forall t) \\
&\phi_{ot}^{*} z_{iot}^{gR} = \bar{z}_{iot}^{gR} \qquad (i = 1,...,n_1 g, \forall t) \\
&\psi_{ot}^{*} z_{iot}^{bR} = \bar{z}_{iot}^{bR} \qquad (i = 1,...,n_1 b, \forall t) \\
&z_{iot}^{bNR} - s_{iot}^{bNR*} = \bar{z}_{iot}^{bNR} \qquad (i = 1,...,n_2 b, \forall t) \\
&z_{iot}^{f} - s_{iot}^{f*} = \bar{z}_{iot}^{f} \qquad (i = 1,...,nf, \forall t) \\
&z_{iot}^{fix} = \bar{z}_{iot}^{fix} \qquad (i = 1,...,nfix, \forall t)
\end{aligned} \tag{6}
$$

Note that radial slacks s_{iot}^{-R*}, s_{iot}^{+R*}, s_{iot}^{gR*} and s_{iot}^{bR*} are not accounted in the above projection. Since they are assumed to be freely disposable and have no effect on efficiency evaluation.

Theorem 1 *The projected DMU* $(\bar{x}_{iot}^{R}, \bar{x}_{iot}^{NR}, \bar{x}_{iot}^{fix}, \bar{y}_{rot}^{R}, \bar{y}_{rot}^{NR}, \bar{y}_{rot}^{fix}, \bar{z}_{iot}^{gR}, \bar{z}_{iot}^{gNR}, \bar{z}_{iot}^{bR}, \bar{z}_{iot}^{bNR}, \bar{z}_{iot}^{f}, \bar{z}_{iot}^{fix})$ *is hybrid-efficient.*

Proof: let an optimal solution of the hybrid model for the projected DMU $(\bar{x}_{iot}, \bar{y}_{rot}, \bar{z}_{iot}^{g}, \bar{z}_{iot}^{b}, \bar{z}_{iot}^{f}, \bar{z}_{iot}^{fix})$ be $(\theta_{ot}^{**}, \phi_{ot}^{**}, \psi_{ot}^{**}, \varphi_{ot}^{**}, \lambda_{o}^{t**}, s_{iot}^{-NR**}, s_{iot}^{+NR**}, s_{iot}^{gNR**}, s_{iot}^{bNR**}, s_{iot}^{f**})$. For proving, it is enough to show that $\theta_{ot}^{**} = \phi_{ot}^{**} = \psi_{ot}^{**} = \varphi_{ot}^{**} = 1$ and all slacks are zero. Since an optimal solution is a feasible solution of the hybrid model, it holds that

$$
\begin{aligned}
&\theta_{ot}^{**} \bar{x}_{iot}^{R} = \theta_{ot}^{**} \theta_{ot}^{*} x_{iot}^{R} \geq \sum_{j=1}^{n} \lambda_{j}^{t**} x_{ijt}^{R} \\
&\bar{x}_{iot}^{NR} = \sum_{j=1}^{n} \lambda_{j}^{t**} x_{ijt}^{NR} + s_{iot}^{-NR**} \quad (A1) \\
&\bar{x}_{iot}^{fix} = \sum_{j=1}^{n} \lambda_{j}^{t**} x_{ijt}^{fix} \\
&\varphi_{ot}^{**} \bar{y}_{rot}^{R} = \varphi_{ot}^{**} \varphi_{ot}^{*} y_{rot}^{R} \leq \sum_{j=1}^{n} \lambda_{j}^{t**} y_{rjt}^{R} \\
&\bar{y}_{rot}^{NR} = \sum_{j=1}^{n} \lambda_{j}^{t**} y_{rot}^{NR} - s_{iot}^{+NR**} \quad (A2) \\
&\bar{y}_{rot}^{fix} = \sum_{j=1}^{n} \lambda_{j}^{t**} y_{rjt}^{fix} \\
&\phi_{ot}^{**} \bar{z}_{iot}^{gR} = \phi_{ot}^{**} \phi_{ot}^{*} z_{iot}^{gR} \leq \sum_{j=1}^{n} \lambda_{j}^{t**} z_{ijt}^{gR}
\end{aligned} \tag{7}
$$

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$$\bar{z}_{iot}^{gNR} = \sum_{j=1}^{n} \lambda_j^{t**} z_{ijt}^{gNR} - s_{iot}^{gNR**} (A3)$$

$$\psi_{ot}^{**} \bar{z}_{iot}^{bR} = \psi_{ot}^{**} \psi_{ot}^{*} z_{iot}^{bR} \le \sum_{j=1}^{n} \lambda_j^{t**} z_{ijt}^{bR}$$

$$\bar{z}_{iot}^{bNR} = \sum_{j=1}^{n} \lambda_j^{t**} z_{ijt}^{bNR} + s_{iot}^{bNR**} (A4)$$

$$\bar{z}_{iot}^{f} = \sum_{j=1}^{n} \lambda_j^{t**} x_{ijt}^{f} + s_{iot}^{f**} (A5)$$

$$\bar{z}_{iot}^{fix} = \sum_{j=1}^{n} \lambda_j^{t**} z_{ijt}^{fix}$$

$$\sum_{j=1}^{n} \lambda_j^{t**} = 1$$

$$\sum_{j=1}^{n} \lambda_j^{t**} z_{ijt}^{\alpha} = \sum_{j=1}^{n} \lambda_j^{t+1**} z_{ijt}^{\alpha}$$

$$\theta_{ot}^{**}, \psi_{ot}^{**} \le 1$$

$$\varphi_{ot}^{**}, \phi_{ot}^{**} \le 1$$

Where $*$ refer to an optimum of the originally stated problem and $**$ refer to an optimum of the problem secured by projection of these solutions to the efficient frontier. According to the model (6), we have

$$A1 \Rightarrow x_{iot}^{NR} = \sum_{j=1}^{n} \lambda_j^{t**} x_{ijt}^{NR} + (s_{iot}^{-NR**} + s_{iot}^{-NR*})$$

$$A2 \Rightarrow y_{rot}^{NR} = \sum_{j=1}^{n} \lambda_j^{t**} y_{rot}^{NR} - (s_{iot}^{+NR**} + s_{iot}^{+NR*})$$

$$A3 \Rightarrow z_{iot}^{gNR} = \sum_{j=1}^{n} \lambda_j^{t**} z_{ijt}^{gNR} - (s_{iot}^{gNR**} + s_{iot}^{gNR*}) \quad (8)$$

$$A4 \Rightarrow z_{iot}^{bNR} = \sum_{j=1}^{n} \lambda_j^{t**} z_{ijt}^{bNR} + (s_{iot}^{bNR**} + s_{iot}^{bNR*})$$

$$A5 \Rightarrow z_{iot}^{f} = \sum_{j=1}^{n} \lambda_j^{t**} z_{ijt}^{f} + (s_{iot}^{f**} + s_{iot}^{f*})$$

With substitution (8) in (7), it can be said that
$(\theta_{ot}^{**}\theta_{ot}^{*}, \phi_{ot}^{**}\phi_{ot}^{*}, \varphi_{ot}^{**}\varphi_{ot}^{*}, \psi_{ot}^{**}\psi_{ot}^{*}, \lambda_o^{t**}, s_{iot}^{-NR**} + s_{iot}^{-NR*}, s_{iot}^{+NR**} + s_{iot}^{+NR*}, s_{iot}^{gNR**} + s_{iot}^{gNR*}, s_{iot}^{bNR**} + s_{iot}^{bNR*}, s_{iot}^{f**} + s_{iot}^{f*})$
is a feasible solution of the hybrid model for DMU_o. From the feasibility of this solution and also the optimality of $\theta_o^{t*}, \psi_o^{t*}, \phi_o^{t*}, \varphi_o^{t*}, \lambda_o^{t*}, s_{ot}^{-NR}*, s_{ot}^{bNR*}, s_{ot}^{+NR*}, s_{ot}^{gNR*}, s_{ot}^{f*}$, the following inequality can be concluded. But firstly, for simplicity of symbols we assume that

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$$\theta_{ot}^{**}\theta_{ot}^{*} = \hat{\theta}_{ot}, \phi_{ot}^{**}\phi_{ot}^{*} = \hat{\phi}_{ot},, \varphi_{ot}^{**}\varphi_{ot}^{*} = \hat{\varphi}_{ot}, \psi_{ot}^{**}\psi_{ot}^{*} = \hat{\psi}_{ot}$$

$$s_{iot}^{-NR**} + s_{iot}^{-NR*} = \hat{s}_{iot}^{-NR}, s_{iot}^{+NR**} + s_{iot}^{+NR*} = \hat{s}_{iot}^{+NR}$$

$$s_{iot}^{gNR**} + s_{iot}^{gNR*} = \hat{s}_{iot}^{gNR}, s_{iot}^{bNR**} + s_{iot}^{bNR*} = \hat{s}_{iot}^{bNR}, s_{iot}^{f**} + s_{iot}^{f*} = \hat{s}_{iot}^{f}$$

Therefore,

$$\frac{\left[1-\frac{m_1}{m}(1-\hat{\theta}_{ot})-\frac{n_1 b}{nb}(1-\hat{\psi}_{ot})-\frac{1}{m}\sum_{i=1}^{m_2}\frac{\hat{s}_{iot}^{-NR*}}{x_{iot}^{NR}}-\frac{1}{nb}\sum_{i=1}^{n_2 b}\frac{\hat{s}_{it}^{bN*R}}{z_{iot}^{bNR}}\right]}{\left[1+\frac{s_1}{s}(\hat{\phi}_{ot}-1)-\frac{n_1 g}{ng}(\hat{\varphi}_{ot}-1)+\frac{1}{s}\sum_{r=1}^{s_2}\frac{\hat{s}_{iot}^{+N*R}}{y_{rot}^{NR}}+\frac{1}{ng}\sum_{i=1}^{n_2 g}\frac{\hat{s}_{iot}^{gNR*}}{z_{iot}^{gNR}}\right]} \geq$$

$$\frac{\left[1-\frac{m_1}{m}(1-\theta_{ot}^{*})-\frac{n_1 b}{nb}(1-\psi_{ot}^{*})-\frac{1}{m}\sum_{i=1}^{m_2}\frac{s_{iot}^{-NR*}}{x_{iot}^{NR}}-\frac{1}{nb}\sum_{i=1}^{n_2 b}\frac{s_{it}^{bN*R}}{z_{iot}^{bNR}}\right]}{\left[1+\frac{s_1}{s}(\phi_{ot}^{*}-1)-\frac{n_1 g}{ng}(\varphi_{ot}^{*}-1)+\frac{1}{s}\sum_{r=1}^{s_2}\frac{s_{iot}^{+N*R}}{y_{rot}^{NR}}+\frac{1}{ng}\sum_{i=1}^{n_2 g}\frac{s_{iot}^{gNR*}}{z_{iot}^{gNR}}\right]}$$

Note that, $\theta_{ot}^{**}, \psi_{ot}^{**} \leq 1$ and $\phi_{ot}^{**}, \varphi_{ot}^{**} \geq 1, s_{iot}^{-NR**}, s_{iot}^{+NR**}, s_{iot}^{bNR**}, s_{iot}^{gNR**} \geq 0$ therefore the above inequality holds if and only if $\theta_{ot}^{**} = \psi_{ot}^{**} = \phi_{ot}^{**} = \varphi_{ot}^{**} = 1$ also $s_{iot}^{-NR**}, s_{iot}^{+NR**}, s_{iot}^{bNR**}, s_{iot}^{gNR**} = 0$. Thus, the projected DMU $(\bar{x}_{iot}, \bar{y}_{rot}\bar{z}_{iot}^{g}, \bar{z}_{iot}^{b}, \bar{z}_{iot}^{f}, \bar{z}_{iot}^{fix})$ is hybrid efficient.

5 Conclusion

In this paper, the dynamic DEA hybrid model was proposed. As a hybrid measure, the suggested model is a combination of radial and non-radial approaches. Also based on dynamic structure, the links or carry-over activities which are categorized into four groups are used in modeling. The proposed model can evaluate the overall efficiency of DMUs for the whole terms as well as the term efficiencies. Finally, the authors of this article are hoped that this study make a small contribution to the future development of dynamic DEA and we look forward to seeing the future research development of the dynamic DEA, as indicated in this study.

References

1. Charnes,A., Cooper, W., Rhodes, E., (1978). Measuring the efficiency of decision making units. European journal of operational research, 2 (6), 429-444.
2. Banker, R., Charnes, A., Cooper, W., (1984). Some models for estimating technical and scale inefficiencies in data envelopment analysis. Management science, 30 (9), 1078-1092.
3. Tone, K., (2001). A slacks-based measure of efficiency in DEA. European Journal of operational Research, 130 (3), 498-509.
4. Charnes, A., Cooper, W., Golany, B., Seiford, L., Stutz, J., (1985). Foundations of data envelopment analysis for pareto-koopmans efficient empirical production functions. Journal of econometrics, 30 (1), 91-107.
5. Russell, R.R., (1985). Measures of technical efficiency. Journal of Economic Theory, 35, 109-126.
6. Pastor, J. T., Ruiz, J. L., Sirvent, L., (1999). An Enhanced DEA Russell Graph Efficiency Measure. European Journal of Operational Research, 115, 596-607.

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7. Tone, K., (2004). A hybrid measure of efficiency in DEA. GRIPS research report series.
8. Klopp, G. A., (1985). The analysis of the efficiency of production system with multiple inputs and outputs. PhD. dissertation, University of Illinois, Industrial and system Engineering College,Chicago.
9. Malmquist, S., (1953). Index numbers and indifference surfaces. Trabajos de Estadistica, 4, 209-242.
10. Fare, R., Grosskopf, S., Norris, S., Zhang, Z., (1994), Productivity growth, technical progress, and efficiency change in industrialized countries.American Economic Review, 84 (1), 66-83.
11. Fare, R., Grosskopf,S., (1996), In:Intertemporal Production Frontiers: With Dynamic DEA. Kluwer, Dordrecht.
12. Tone, K., Tsutsui, M., (2009). Dynamic DEA: A slacks-based measure approach. Omega 38, 145-156.
13. Amirteimori, A. R., (2006). Data envelopment analysis in dynamic framework. Applied Mathematics and Computation, 181, 21-28.
14. Chen, C. M., (2009). Network-DEA. A model with new efficiency measures to incorporate the dynamic effect in production network. European Journal of Operational Research, 194 (3), 687-699.
15. Nemoto, J., Goto, M., (1999). Dynamic DEA: modeling inter -temporal behavior of a firm in the presence of productive inefficiencies. Economics Letters, 64, 51-56.
16. Nemoto, J., Goto, M., (2003). Measuring dynamic efficiency in production: an application of DEA to Japanese electric utilities. Journal of Productivity Analysis, 19 (2-3), 191-210.
17. Park, K. S., Park, K., (2005). Measurement of multi -period aggregative efficiency. European Journal of Operational Research, 193 (2), 567-580.
18. Sueyoshi, T., Sekitani, K., (2005). Returns to scale in dynamic DEA. European Journal of Operational Research, 161 (2), 536-544.
19. Chang, H., Choy, H.L., Cooper, W. W., Ruefli, W. T., (2009). Using Malmquist indexes to measure changes in the productivity and efficiency of US accounting firms before and after the Sarbanes-Oxley Act Omega, 37 (5), 951-960.
20. Sanei, M., Shahsavan, T., Fallah Jelodar, M., (2008). Dynamic DEA and its development for additive model to evaluate return to scale. Applied Mathematical Sciences, 30 (2), 1479-1486.

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